Synthesis of Plane Linkages With Use of the Displacement Matrix

A generalized matrix for the description of rigid body displacement in two dimensions is developed. This displacement matrix is applied to the synthesis of plane linkages used for rigid body guidance, path generation, and function generation.

1 Introduction

Beginning in 1955 with the work of F. Freudenstein and his co-workers [1, 2], there has been an extensive development of analytical methods for the synthesis of plane mechanisms which make efficient use of the capabilities of modern digital computers. Major emphasis has been placed on the use of complex polar vector notation to describe relative positions in the linkage of an assumed type from which analytical expressions for relative displacements, velocities, or accelerations can be determined. These equations are then combined with design conditions to give a set of simultaneous equations to be solved for unknown mechanism parameters.

A second and somewhat different design method is the synthesis of linkages directly from specified finite displacements given in numerical form. Wilson [3] has developed design equations which locate center-point and circle-point curves of plane mechanisms in terms of the classic rotation matrix operator combined with the displacement of a point in the moving body initially located at the origin of the coordinate system. The rotation matrix is somewhat limited in its application since all rotations must be specified about axes passing through the origin of the coordinate system.

The present paper gives an extension of the finite displacement method for plane mechanism synthesis using a 3 × 3 displacement matrix operator, which allows a more generalized description of plane displacement than the usual 2 × 2 plane rotation matrix. A future paper will describe the application of the 4 × 4 displacement matrix operator to problems in the synthesis of three-dimensional linkages.

2 Geometric Transformations

Geometric transformations [4] are a part of the mathematical notion of function. The simplest type of geometric transformation is the point transformation, in which every point considered as an element of one space is transformed into a corresponding point in a second space. A particularly simple group of geometric transformations, useful in kinematics, is the affine transformation, in which points located on a straight line in one space are transformed into corresponding points on a straight line in a second space.

An n-dimensional affine space \( \mathbb{A}^n \) is a set of elements (points) having a one-to-one mapping onto the n-dimensional vector space \( \mathbb{V}^n \). An affine transformation in two-dimensional space is defined analytically when \( x, y \) are linear functions of \( \alpha, \beta \). Expressed as a homogeneous matrix equation,

\[
\begin{bmatrix}
\alpha \\
\beta \\
1
\end{bmatrix}
= \begin{bmatrix}
1 & a & b \\
0 & 1 & c \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
1
\end{bmatrix}
\]

Rigid body displacement without distortion can be considered as a special case of an affine transformation.

3 Displacement: Mathematical Description

A particular rigid body displacement, as defined above, can be considered as one element of the group which consists of the system \( \mathbb{D} \) of all rigid body displacements. We let \( [D] \) in matrix form represent an element of the system \( \mathbb{D} \) and adopt matrix multiplication as the rule of combination of the group.

From the mathematical definition of a group [5] we may state the following conditions which must be satisfied by displacement matrices:

(a) The product of any two of the system of generalized displacement matrices is also a displacement matrix which forms an element of the same system in Euclidean space.

\[ [D_1][D_2] = [D] \]

(b) The product of any three displacement matrices is associative.

\[ ([D_1][D_2])[D_3] = [D_1]([D_2][D_3]) \]

(c) There exists an identity matrix \([I]\) which is an element of the system.

(d) For any displacement matrix \([D_1]\) in the system, there exists an inverse \([D_1]^{-1}\) in the same system such that

\[ [D_1][D_1]^{-1} = [I] \]

4 Basic Displacement Matrices

After establishment of the mathematical and geometric basis for the general two-dimensional displacement matrix, it is important to examine the analytical form of the displacement matrix under certain conditions. Displacement matrices will be formulated as combinations of translation and rotation about the origin of the coordinate system, in order to display the components of
the matrix analytically in a form which will allow the resolution of a numerical matrix into its components.

**Translation in Two Dimensions**

Refer to Fig. 1, which indicates a translation displacement of a plane on a plane where each point in the plane undergoes a displacement $\Delta x$, $\Delta y$. Under these conditions

$$[D_{1}]_T = \begin{bmatrix} 1 & 0 & \Delta x \\ 0 & 1 & \Delta y \\ 0 & 0 & 1 \end{bmatrix}$$  \tag{5}

where

$$\Delta x = x_2 - x_0, \quad \Delta y = y_2 - y_1$$

**Rotation in Two Dimensions About the Origin**

Fig. 2 illustrates the rotation of a plane containing a point $P$ about a fixed origin of the coordinate system. The rotation matrix takes the familiar form,

$$[D_{2}]_R = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$  \tag{6}

**General Motion in Two Dimensions**

The displacement matrix for general plane motion can be formulated in several ways. Assume a rotation pole $A_1(x_0, y_0)$ exists which forms a center of rotation in the plane. First, form a displacement matrix $[D_{1}]_T$ such that the pole $A_1(x_0, y_0)$ will coincide with the origin.

$$[D_{1}]_T = \begin{bmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{bmatrix}$$  \tag{7}

Since

$$\Delta x = 0 - x_0 = -x_0, \quad \Delta y = 0 - y_0 = -y_0$$

Second, allow the rotation displacement $[D_{2}]_R$ about the origin $(0, 0)$

$$[D_{2}]_R = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$  \tag{8}

Finally, translate the point $A_1$ back to its original position.

$$[D_{2}]_T = \begin{bmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{bmatrix}$$  \tag{9}

Therefore,

$$[D_{1}] = [D_{2}]_T [D_{3}]_R [D_{1}]_T$$

$$[D_{1}] = \begin{bmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{bmatrix}$$  \tag{10}

Since

$$A_1' = \begin{bmatrix} x_1' \\ y_1' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}$$

we obtain

$$[D_{2}]_R = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$  \tag{11}

Fig. 3 illustrates two-dimensional motion in which the position of the rotation pole is unknown. First rotate $A_1B_1$ about the origin through the angle $\theta$ to position $A_1'B_1'$. The line $A_1'B_1'$ is then moved to position $A_2'B_2$ by direct translation.

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Comparing (10) and (11), we note that the rotation pole
\( P(x_0, y_0) \) may be determined from \( A_2(x_0, y_0) \), \( A_2(x_0, y_0) \), and \( \theta \) by
equating corresponding elements of the displacement matrix. A
third form would be expressed in terms of the coordinates of
specific points of the moving plane in the initial and final
positions. Three points, \( A, B \), and \( C \), are specified in positions
\( A(x_1, y_1) \), \( B(x_2, y_2) \), and \( C(x_3, y_3) \) and in positions \( A(x_1, y_1) \),
\( B(x_2, y_2) \), and \( C(x_3, y_3) \). The \( [D_3] \) matrix may be found by
noting that

\[
\begin{bmatrix}
  x_1 & x_2 & x_3 \\
  y_1 & y_2 & y_3
\end{bmatrix}
= [D_3]
\begin{bmatrix}
  x_1' & x_2' & x_3' \\
  y_1' & y_2' & y_3'
\end{bmatrix}
\]

Therefore

\[
[D_3] = \begin{bmatrix}
  x_1 & x_2 & x_3 \\
  y_1 & y_2 & y_3 \\
  1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
  x_1' & x_2' & x_3' \\
  y_1' & y_2' & y_3' \\
  1 & 1 & 1
\end{bmatrix}
\]

This leads to a fourth expression similar to equation (12) but
involving two moving points only.

\[
[D_3] = \begin{bmatrix}
  x_1 & x_2 & x_3 \\
  y_1 & y_2 & y_3 \\
  1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
  x_1' & x_2' & x_3' \\
  y_1' & y_2' & y_3' \\
  1 & 1 & 1
\end{bmatrix}^{-1}
\]

Any of the forms given in equations (10), (11), (12), and (13)
will result in a \( 3 \times 3 \) numerical form of the

\[
[D_3] = \begin{bmatrix}
  A_{11} & A_{12} & A_{13} \\
  A_{21} & A_{22} & A_{23} \\
  0 & 0 & 1
\end{bmatrix}
\]

The matrix equation

\[
\begin{bmatrix}
  X_1 \\
  Y_1 \\
  1
\end{bmatrix}
= [D_3]
\begin{bmatrix}
  X_1 \\
  Y_1 \\
  1
\end{bmatrix}
\]

represents a coordinate transformation in which any point of the
set which comprises the moving plane is transformed from position
1 into position 2.

5 General Method of Synthesis

The basic problem in the synthesis of plane linkages is to locate
those points of the moving plane which, as the plane assumes
specified positions, assume a series of positions that lie on a cir-
cular arc. These particular points, designated circle points, can be
used as hinge points in the moving plane. Two lines, each with
one end connected to the moving hinge points (circle points) and
the other end to ground at the center of the corresponding circular
arc (center points), will guide the plane through the specified
positions.

Assume that a plane displacement is characterized in the general
form of a \( [D] \) matrix similar to equation (11).

Let \( X_1 \) and \( Y_1 \) be the unknown coordinates of a moving pivot
in position 1. The \( n \)th position of the moving pivot is expressed in
terms of the first position as

\[
\begin{bmatrix}
  X_n \\
  Y_n \\
  1
\end{bmatrix}
= [D_n]
\begin{bmatrix}
  X_1 \\
  Y_1 \\
  1
\end{bmatrix}
\]

If \( (X_n, Y_n) \) are the coordinates of the corresponding fixed pivot,
the condition of constant link length requires that

\[
(X_i - X_j)^2 + (Y_i - Y_j)^2 = (X_n - X_0)^2 + (Y_n - Y_0)^2
\]

Squaring expressions for \( X_n \) and \( Y_n \) found from equation (16)
and collecting terms leads to

\[
X_1(A_{12} \cos \theta_{12} - A_{13} \sin \theta_{12}) - A_{14} \cos \theta_{12} - A_{15} \sin \theta_{12} + X_n = X_2 \cos \theta_{12} - Y_2 \sin \theta_{12} + Y_n
\]

Example Problem 1

\[
(X_1, Y_1) = (3, 1.5) \quad \theta_{12} = 45 \text{ deg}
\]

\[
A_1 = A_2(x_0, y_0) = (1, 1)
\]

\[
A_1 = A_2(x_0, y_0) = (2, 0.5) \quad \theta_{12} = 0 \text{ deg}
\]

\[
A_1 = A_2(x_0, y_0) = (3, 1.5) \quad \theta_{12} = 45 \text{ deg}
\]

A four-bar linkage will be designed with the fixed pivot for one
crank located at \( B_0(X_0, Y_0) = B_0(0, 0) \) and the fixed pivot for the
second crank located at \( C_0(X_0', Y_0') = C_0(5, 0) \).
Numerical solution is by desk calculator to six decimal point accuracy.

Substituting into equation (18) \((x_1, y_1) = (1, 1), (x_2, y_2) = (2, 0.5), (X_0, Y_0) = (0, 0),\) and \(\theta_0 = 0,\) we have

\[
X_1[1 + (-0.5)(0)] + Y_1[-0.5 - 0] = -\frac{1}{2}[1 + (-0.5)]^2
\]

that is,

\[
X_1 - 0.5Y_1 = -0.625000 \quad (21)
\]

Similar substitutions for the displacement to position 3 lead to

\[
2.181981 A_1' - 2.060661 A_1 = -4.503680 \quad (22)
\]

Solving (21) and (22) simultaneously we obtain

\[
X_1 = 0.994078 \\
Y_1 = 3.238155 
\]

i.e., \(A_1 = A_1(0.994078, 3.238155)\)

\[
B \equiv B(0, 0)
\]

Similar substitutions for an assumed pivot \(C_0(5, 0)\) lead to a pair of equations in \(X_1'\) and \(Y_1'.\)

\[
X_1' - 0.5Y_1' = 4.375000 \quad (23) \\
3.646446X_1' + 1.474874Y_1' = 10.496320 \quad (24)
\]

which give the coordinates of the second moving pivot in its first position:

\[
C_1 = C_1(3.547725, -1.654550) \\
C_0 = C_0(5, 0)
\]

The completed mechanism is shown in Fig. 4.

**Example Problem 2**

The conditions of Example Problem 1 are repeated. In this case, however, the guidance is to be accomplished by a slider-crank mechanism such that the fixed crank pivot is located at \(C_0 = C_0(5, 0)\) and the slider is located somewhere on the \(y\)-axis \((x = 0)\) in its first position.

The choice of fixed crank pivot at \(C_0 = C_0(5, 0)\) has already been shown to result in a moving pivot \(C_1 = C_1(3.547725, -1.654550)\) and the same crank \(C_0C_1\) will be used in this design. Since for the slider, \(X_1 = 0\), we need only to determine the value of \(Y_1\).

From equation (11),

\[
[D_{13}] = \begin{bmatrix} 0.707107 & -0.707107 & 3 \\ 0 & 0.707107 & 0.085786 \\ 0 & 0 & 1 \end{bmatrix}
\]

we have

\[
\begin{bmatrix} X_2 \\ Y_2 \\ Y_1 \end{bmatrix} = [D_{13}] \begin{bmatrix} 0 \\ Y_1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ Y_1 - 0.5 \end{bmatrix}
\]

\[
\begin{bmatrix} X_2 \\ Y_2 \\ Y_1 \end{bmatrix} = [D_{13}] \begin{bmatrix} 0 \\ Y_1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.707107Y_1 + 3 \\ 0.707107Y_1 + 0.085786 \\ 1 \end{bmatrix}
\]

From

\[
\begin{bmatrix} X_1 \\ Y_1 \\ 1 \end{bmatrix} = 0 \\
\begin{bmatrix} X_3 \\ Y_2 \\ 1 \end{bmatrix} = 0 \\
\begin{bmatrix} X_2 \\ Y_2 \\ 1 \end{bmatrix} = 0
\]

we have \(Y_1 = 2.453100\). That is, the slider should be located so that in its first position the slider pivot has the coordinates \(D_1 = D_1(0, 2.453100)\). The slope of the slider path is

\[
m = \tan \theta = \frac{Y_3 - Y_1}{X_3 - X_1} = \frac{Y_3 - Y_2}{X_3 - X_2} = -0.5
\]

\[
\theta = \tan^{-1}(-0.5) = -26.565°
\]

The completed mechanism is shown in Fig. 5.

**7 The Four-Position Guidance Problem**

**Example Problem 3**

A four-position guidance problem has been created, as shown in Fig. 7, by adding a fourth position to the three-position guidance problem of Example Problem 1. Here again, the fact that the first two positions of the member are parallel would cause difficulty in the use of Burmester theory for the graphical determination of the center-point and circle-point curves. In this problem, the four positions of the plane are again specified in terms of the displacement of one point \((A)\) in the plane and the associated angular rotation of the plane.

\[
A_1 = A_1(x_0, y_1) = (1, 1) \\
A_2 = A_1(x_0, y_2) = (2, 0.5) \\
\theta_{13} = 0
\]
A3 = A3(x, y) = (3, 1.5) \quad \theta_0 = 45 \text{ deg}
A4 = A4(x, y) = (2, 2) \quad \theta_4 = 90 \text{ deg}

[D_{A3}] \text{ and } [D_{A4}] \text{ have been specified previously in Example Problem 2.}

\begin{align*}
[D_{A3}] &= \begin{bmatrix}
0 & -1 & 3 \\
1 & 0 & 1 \\
0 & 0 & 1
\end{bmatrix}
\end{align*}

Again, letting \((X_1, Y_1)\) be the first position of a moving pivot and \((X_2, Y_2)\) its associated fixed pivot, we may write three matrix equations for the displacements of the moving pivot \((X, Y)\):

\begin{align*}
\begin{bmatrix}
X_2 \\
Y_2
\end{bmatrix} &= [D_{A3}] \begin{bmatrix}
X_1 \\
Y_1
\end{bmatrix} = \begin{bmatrix}
X_1 + 1 \\
Y_1 - 0.5
\end{bmatrix}
\end{align*}

\begin{align*}
\begin{bmatrix}
X_3 \\
Y_3
\end{bmatrix} &= [D_{A4}] \begin{bmatrix}
X_1 \\
Y_1
\end{bmatrix} = \begin{bmatrix}
0.707107X_1 - 0.707107Y_1 + 3 \\
0.707107X_1 + 0.707107Y_1 + 0.085786
\end{bmatrix}
\end{align*}

\begin{align*}
\begin{bmatrix}
X_4 \\
Y_4
\end{bmatrix} &= [D_{A4}] \begin{bmatrix}
X_1 \\
Y_1
\end{bmatrix} = \begin{bmatrix}
-Y_1 + 3 \\
X_1 + 1
\end{bmatrix}
\end{align*}

To insure constant length of the guiding link we have three circle equations,

\begin{align*}
(X_1 - X_3)^2 + (Y_1 - Y_3)^2 &= (X_2 - X_3)^2 + (Y_2 - Y_3)^2 \\
(X_3 - X_4)^2 + (Y_3 - Y_4)^2 &= (X_4 - X_5)^2 + (Y_4 - Y_5)^2
\end{align*}
The colinearity of points \( A_1, A_2, A_3, \) and \( A_4 \) is specified by the two straight-line equations
\[
\begin{align*}
X_1 Y_1 & \begin{pmatrix} X_2 & Y_2 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} = 0 \\
X_2 Y_2 & \begin{pmatrix} X_3 & Y_3 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} = 0 \\
X_3 Y_3 & \begin{pmatrix} X_4 & Y_4 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} = 0
\end{align*}
\]
Expanding the determinants, we have
\[
\begin{align*}
Y_2 - Y_1 & = \frac{Y_4 - Y_1}{X_4 - X_1} \\
X_2 - X_1 & = \frac{X_4 - X_1}{X_4 - X_1}
\end{align*}
\]
The solution of equations (31) is carried out as part of the computer program for four-position plane mechanism synthesis. The computer program gave the following results:
\[
A = A(-1.472791, 1.175736)
\]
\[
\theta = \tan^{-1}(-0.5) = -26^\circ 35' \cos \theta
\]
\[
\sin \theta
\]

8 Design of Four-Bar Function Generators

Consider the problem of designing a four-bar linkage such that the input-output crank motions are proportional to a specified functional relationship between two variables at a given number of precision points.

The displacement matrix for the relative motion of the input crank with respect to the output crank may be developed by separation of the total relative motion into its components. The input crank has its fixed center \( A_0 = A_0(0, 0) \) located at the origin. The fixed center for the output crank is located at \( B_0 = B_0(1, 0) \). The total relative motion is composed of a rotation +\( \theta \) of the input crank followed by a rotation −\( \phi \) about the output crank center. The −\( \phi \) component results in an inversion of the entire mechanism about the first position of the output crank.

In those cases where both cranks rotate in the same direction,
\[
[D_R]_{1R} = [D]_{1-R}[D]_{1R}
\]

These equations correspond to equation (11) with \( \theta = (\theta_n - \phi_n) \), \( \phi = (\phi_n - \phi_n) \), and \( \theta_n = (1 - \cos \phi_n \sin \phi_n) \).

When it is desired to have \( \theta \) and \( \phi \) with opposite directions of rotation, i.e., in a crossed linkage, \( \phi_n \) is replaced by −\( \phi_n \) in equation (32), with the result
\[
[D_R]_{1-R} = \begin{pmatrix}
\cos (\theta_n - \phi_n) & \sin (\theta_n - \phi_n) & 1 - \cos \phi_n \\
\sin (\theta_n - \phi_n) & \cos (\theta_n - \phi_n) & \sin \phi_n
\end{pmatrix}
\]

Example Problem 5

Design of a four-bar function generator with four precision positions and velocity ratio specified in the first of the four positions.

The design will be arbitrarily assumed to be of the crossed-linkage type.

Function: \( y = e^x \)
Range: \( 0 \leq x \leq 1.2 \)
Input angle: \( 0 \leq \theta \leq -90 \text{ deg} \) (clockwise)
Output angle: \( 0 \leq \phi \leq +90 \text{ deg} \)
Precision points: \( x = 0, 0.4, 0.8, 1.2 \)
Velocity ratio: \( VR = -1 \) at \( x = 0 \) corresponding to velocity pole at \( (0.5, 0) \) when linkage is in first position.

The results of the computer calculations are plotted in Fig. 9 in the form of the loci of the possible moving pivot \( A_1 \) and the asso-
9 Design of Four-Bar Path Generators

The design of a four-bar mechanism for the guidance of a point \( P_1(a_1, b_1) \) through a series of points \( P_n(a_n, b_n) \) on a given path is accomplished by the iterative solution of a set of simultaneous nonlinear equations involving the unknown parameters \( p_n, q_n, p_1, q_1, r_n, s_n, a_n, \) and \( b_n, \) as shown in Fig. 10. Since the rotation angle \( \theta_{in} \) is listed as an unknown, a new variable will be added for each additional precision point \( P_n \) specified along the path. This results in the advantage of either increasing the number of specified path precision points or, alternatively, making possible greater freedom in the arbitrary specification of design parameters. By comparison, in the design of rigid body guidance mechanisms, the coupler rotation angles \( \theta_{in} \) are specified as design conditions.

For each position \( n \) there are two design equations based on the constraints imposed by constant length for both links \( L_s \) and \( L_t.\)

\[
(L_t)^2 = (p_t - p_0)^2 + (q_t - q_0)^2 = (p_t - p_0)^2 + (q_t - q_0)^2
\]

\[
(L_s)^2 = (r_s - r_0)^2 + (s_s - s_0)^2 = (r_s - r_0)^2 + (s_s - s_0)^2
\]

(34)

Since

\[
\begin{bmatrix}
 p_n \\
 q_n \\
 1
\end{bmatrix} = [D_{int}]
\begin{bmatrix}
 p_1 \\
 q_1 \\
 1
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
 r_s \\
 s_s \\
 1
\end{bmatrix} = [D_{int}]
\begin{bmatrix}
 r_1 \\
 s_1 \\
 1
\end{bmatrix}
\]

where \([D_{int}]\) is the form of equation (11) with \( x_i = a_i, x_2 = a_n, y_1 = b_1, \) and \( y_2 = b_n, \) it can be shown that equations (34) lead to a pair of design equations suitable for computer programming of the form,

\[
\begin{aligned}
(p_t - p_0) + a_0p_t - b_0q_t - a_1p_t - b_1q_t + a_1^2 + b_1^2 + a_n^2 + b_n^2 \\
+ \cos \theta_{int}[-p_{int} - q_{int}] + a_{int} + b_{int}
\end{aligned}
\]

\[
- a_0b_t + b_0a_t + a_1b_t + b_1a_t
\]

\[
+ \sin \theta_{int}[p_{int} - q_{int}] - b_{int} + a_{int}
\]

\[
[+ a_0b_t - a_0a_t + b_0b_t - a_{int} = 0]
\]

(Plus a similar expression from the second of equations (34)/(35)

\[
\quad (L_s)^2 = (p_t - p_0)^2 + (q_t - q_0)^2
\]

(36)

In the case of five precision point path generation, there would be 10 equations and 12 unknowns, and two parameters may be specified, e.g., \( p_0, q_0. \)

Example Problem 6

Design of a path generation mechanism with five path precision points.

Fig. 11 illustrates a four-bar mechanism for the guidance of a point \( P \) through precision points.

\[
P_1(a_1, b_1) = (1.00000, 1.00000)
\]

\[
P_2(a_2, b_2) = (2.00000, 0.50000)
\]

\[
P_3(a_3, b_3) = (3.00000, 1.50000)
\]

\[
P_4(a_4, b_4) = (2.00000, 2.00000)
\]

\[
P_5(a_5, b_5) = (1.50000, 1.90000)
\]

In this example, fixed pivots were initially assumed at.

\[
A_3(p_0, q_0) = (2.10000, 0.60000)
\]

\[
B_4(q_0, s_0) = (1.30000, 4.20000)
\]

1967 Transactions of the ASME

Table 1 Parametric relationships—four-bar path generator

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<th>No. of Variables</th>
<th>No. of Specified Variables</th>
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<tr>
<td>Possible unique solution</td>
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</table>

*Symbols represent coordinates as shown in Figure 10.

1Solutions highly dependent upon initial guesses and numerical accuracy.

The first position. This specification would add the following two equations to equations (35).

\[
(L_t)^2 = (p_t - p_0)^2 + (q_t - q_0)^2
\]

\[
(L_s)^2 = (r_t - r_0)^2 + (s_t - s_0)^2
\]

Transactions of the ASME

Transactions of the ASME

Transactions of the ASME

Transactions of the ASME
The computer solution gave
\[ A_1(p_1, q_1) = (0.6073749, -1.127103) \]
\[ B_1(r_1, s_1) = (-0.5863996, 0.9969990) \]

The computer program will also determine the loci of possible moving pivots \( A_1 \) and \( B_1 \) for arbitrary finite changes in the location of either fixed pivot. These loci are shown in Fig. 11 for 13 positions of \( B_0(r_0, s_0) \) spaced uniformly on a straight line between \( B_0(1.50000, 4.20000) \) and \( B_0(-1.50000, 3.00000) \).

**Example Problem 7**

Design of a path generation mechanism with the same five path precision points of Example Problem 6 but with fixed pivot specified only at
\[ A_2(p_2, q_2) = (2.10000, 0.50000) \]

Two cranks of arbitrary length are specified as \( L_2 = 1.0 \) unit and \( L_4 = 2.0 \) units. The computer solution gave
\[ B_0(r_0, s_0) = (0.6934239, 1.184073) \]
\[ A_1(p_1, q_1) = (1.206753, 0.05043468) \]
\[ B_1(r_1, s_1) = (0.3341094, -0.7833851) \]

The computer program will also determine the loci of possible moving pivots \( A_1 \) and \( B_1 \) and of one fixed pivot \( B_0 \) for arbitrary finite changes in the location of the assumed fixed pivot \( A_2 \). These loci are shown in Fig. 12 for seven positions of \( A_2(p_2, q_2) \) spaced uniformly on a vertical straight line between \( A_2(2.10000, 0.50000) \) and \( A_2(2.10000, 1.10000) \).

### 10 Design of Geared Five-Bar Function Generators

Sandor [2] has discussed the use of geared five-bar linkages as function generators and has shown the possibility of a six-precision-point solution with specified gear ratio obtained by the complex-number method.

With use of the displacement matrix methods of the present paper, it is possible to develop a seven-precision-point solution for an arbitrarily assumed gear ratio \( R \). A maximum of eight precision points is theoretically possible if \( R \) is considered as a variable in the equations. The practical realization of such a solution may be difficult when gears with finite number of teeth are used. Use of crossed belts or friction wheels would be a possibility.

The design equation is developed by considering the motion of link 4 relative to link 2, as shown in Fig. 13. The displacement matrix for link 4 is formed by considering successive rotations about points \( B_0, B_1, \) and \( A_2 \); this procedure leads to equation (37).

\[
[D_{41}][\theta] = \begin{bmatrix}
\cos \beta_{in} & -\sin \beta_{in} & r_1(\cos \alpha_{in} - \cos \beta_{in}) - \delta_2(\sin \alpha_{in} - \sin \beta_{1m}) + (1 - \cos \phi_{im}) \\
\sin \beta_{in} & \cos \beta_{in} & r_1(\sin \alpha_{in} - \sin \beta_{in}) + \delta_2(\cos \alpha_{in} - \cos \beta_{in}) + \sin \phi_{im} \\
0 & 0 & 1
\end{bmatrix}
\]

where
\[
\alpha_{in} = (\theta_{in} - \phi_{im}) \\
\beta_{in} = (\theta_{in} - \phi_{im} + R\theta_{im})
\]

From the condition \( L_3 = \text{constant} \) there will be \((n - 1)\) design equations which may be expressed in a form suitable for programming as,
\[
(u_i - p_i) + (v_i - q_i) = \left[ C_{1i}u_i - D_{1i}v_i + r_i(A_{1i} - C_{1i}) - \delta_2(B_{1i} - D_{1i}) + E_{1i} - p_i \right]^2 \\
+ [D_{1i}u_i + C_{1i}v_i + r_i(B_{1i} - D_{1i}) + \delta_2(A_{1i} - C_{1i}) + F_{1i} - q_i]^2
\]

**Fig. 11** Example Problem 6: Four-bar linkage for path generation with five precision points and arbitrary choice of both fixed pivots. One fixed pivot varied to display loci of two moving pivots.

**Fig. 12** Example Problem 7: Four-bar linkage for path generation with five precision points. Arbitrary specification of one fixed pivot and lengths of both guiding cranks. One fixed pivot varied to display loci of two moving pivots and second fixed pivot.

**Fig. 13** Example Problem 8: Geared five-bar function generator with seven precision points. Gear ratio arbitrarily specified as +0.50.
where
\[ A_{in} = \cos \alpha_{in} \]
\[ B_{in} = \sin \alpha_{in} \]
\[ C_{in} = \cos \beta_{in} \]
\[ D_{in} = \sin \beta_{in} \]
\[ E_{in} = 1 - \cos \phi_{in} \]
\[ F_{in} = \sin \phi_{in} \]

(38)

Table 2 gives an analysis of the parametric relationships for geared five-bar function generators and indicates the possibility of a maximum of eight precision points for the geared five-bar linkage. A seven-point solution is presented as an example.

Example Problem 8

Design of a geared five-bar function generator with seven precision points with arbitrary gear ratio \( R = 0.5 \).

Input and output angles were assumed according to the following schedule.

\[ \theta_2 = 10.0 \text{ deg} \]
\[ \phi_2 = 20.0 \text{ deg} \]
\[ \theta_3 = 18.0 \text{ deg} \]
\[ \phi_3 = 34.5 \text{ deg} \]
\[ \theta_4 = 20.0 \text{ deg} \]
\[ \phi_4 = 38.0 \text{ deg} \]
\[ \theta_5 = 30.0 \text{ deg} \]
\[ \phi_5 = 55.0 \text{ deg} \]
\[ \theta_6 = 33.0 \text{ deg} \]
\[ \phi_6 = 60.0 \text{ deg} \]
\[ \theta_7 = 36.0 \text{ deg} \]
\[ \phi_7 = 65.0 \text{ deg} \]

The parameters of the resulting mechanism, found by the computer program, are

\[ \gamma_1 = 1.042952 \]
\[ \psi_1 = -0.1362623 \]
\[ \mu_1 = 0.3890990 \]
\[ \nu_1 = -0.2806455 \]
\[ \pi_1 = 0.4497287 \]
\[ \delta_1 = -0.3728327 \]

The solution is shown in Fig. 13 with \( R = 0.5 \). It should be noted that there are no restrictions on the choice of gear ratio \( R \) except as dictated by the practical choice of number of teeth on mating gears. A number of gear ratios could be investigated simultaneously by plotting loci of moving pivot locations as a function of assumed gear ratio.

Using the methods of the present paper, the authors have also obtained solutions for path generation mechanisms of the two-gear five-bar and the four-gear six-bar types. In these cases also, solutions may be obtained for an arbitrary choice of gear ratio with corresponding reduction in number of path precision points.

11 Conclusion

The numerical methods based upon the displacement matrix have proved to be useful in the synthesis of any type of plane mechanism constrained by lower pairs. The method is suitable for desk calculator computation with up to three precision points and makes efficient use of digital computers for larger numbers of precision points.

The present work is a part of a study of mathematical methods useful in the design of three-dimensional exoskeletal systems for the human body, particularly orthopaedic braces for the lower extremity. The support of the Veterans Administration, through Contract No. V100SM-2075 with the Prosthetic and Sensory Aids Service, is gratefully acknowledged.

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References