1 Introduction

Previous work in the synthesis of spherical linkages has been largely concerned with the problem of the design of spherical linkage function generators. Beyer [1] has described and given references for a number of graphical and analytical methods developed in Europe. In the United States the matrix methods of Denavit and Hartenberg [2] have been used to derive design equations for function generators in terms of linkage parameters and input and output angles in a form similar to the Denavit and Hartenberg equation for plane synthesis. Wilson [3] has developed methods based upon equation (3) of the present paper for the numerical synthesis of function generators. Johnson [4] has shown a graphical-analytical method for the approximate synthesis of three-dimensional path generation mechanisms.

Methods of analysis for spherical linkages have been developed extensively in recent years. Chase [5] has summarized the methods of vector calculus for three-dimensional problems, Hartenberg and Denavit [2] have shown the application of matrix methods, and Yang [6, 7] has developed the use of quaternion algebra, including application to force and torque analysis.

The present paper is concerned with the development of a general method for the synthesis of spherical linkages. Emphasis is placed upon the problem of rigid body guidance with function generation considered as a special case solved by kinematic inversion. A maximum of five precision positions of the moving body can be specified.

A second class of problems involves the specification of a number of precision points on a general three-dimensional path in space. Any four points in space can be shown to lie on a sphere, hence a maximum of four precision points may be generated by a coupler point of a spherical linkage in the general case. A maximum of nine path precision points is possible with a spherical linkage if all points lie on the surface of a sphere.

2 Displacement Matrix

Spherical rigid body motion is defined as a special case of three-dimensional motion in which all moving points are constrained to move on a spherical surface; i.e., all moving points maintain a constant distance from a fixed reference point. Spherical motion is similar to plane motion in the sense that they are both degenerate cases of general three-dimensional motion with a decrease in the number of parameters required to describe a rigid body displacement.

3 Coordinate Systems for Spherical Motion

Spherical displacement of a point is often described in terms of the radius of the spherical surface and a change in two spherical angles. In the present discussion of methods for the synthesis of spherical linkages, it is convenient to give the coordinates of precision points as Cartesian coordinates, and then to calculate the coordinates of the center and the radius of a spherical surface containing these points. These coordinates are then normalized such that the Cartesian coordinates of any point \( P(x, y, z) \) on the sphere must satisfy the relationship \( x^2 + y^2 + z^2 = 1 \). The direction cosines of an axis through a point \( P(x, y, z) \) are given directly by the coordinates, i.e., \( u_x = x, u_y = y, u_z = z \). The angle \( \alpha \) between axes through two points \( P(x_1, y_1, z_1) \) and \( P(x_2, y_2, z_2) \) on the sphere is easily determined from the relation,

\[
\cos \alpha = x_1 x_2 + y_1 y_2 + z_1 z_2 \tag{1}
\]

4 Displacement Matrices for Spherical Rigid Body Motion

Rotation Matrix in Analytical Form

Assume a spherical rigid body motion about an axis \( u \) with direction cosines \( u_x, u_y, u_z \). A point \( P_1 = P(x_1, y_1, z_1) \) on a rigid body is displaced to a position \( P_2 = P(x_2, y_2, z_2) \) by a rotation \( \phi \) about the given axis \( u \). The spherical displacement may be expressed as a vector equation [3, 8, 9, 10, 11] in the form,

\[
\begin{bmatrix}
\begin{array}{c}
x' \\ y' \\ z'
\end{array}
\end{bmatrix} = \begin{bmatrix}
\begin{array}{ccc}
x_x & y_x & z_x \\ x_y & y_y & z_y \\ x_z & y_z & z_z
\end{array}
\end{bmatrix} \begin{bmatrix}
\begin{array}{c}
x \\ y \\ z
\end{array}
\end{bmatrix} + \begin{bmatrix}
\begin{array}{c}
x_0 \\ y_0 \\ z_0
\end{array}
\end{bmatrix}
\]

where

\[ r' = \begin{bmatrix}
\begin{array}{c}
x' \\ y' \\ z'
\end{array}
\end{bmatrix} \]

\[ r = \begin{bmatrix}
\begin{array}{c}
x \\ y \\ z
\end{array}
\end{bmatrix} \]

This vector equation may be expressed in terms of its scalar components as a rotation matrix equation \([r'] = [R] [r] \).
situations, particularly in motion analysis, it lacks sufficient generality to be useful in numerical synthesis.

**General Rigid Body Motion in Space**

As discussed in a previous paper [12], a rigid body displacement is defined mathematically as a special case of an affine transformation in which all points of the body maintain a fixed distance between each other after the transformation. An affine transformation in three dimensions is a linear transformation of the form:

\[
\begin{align*}
x &= a_{11}x + a_{12}y + a_{13}z + a_{14} \\
y &= a_{21}x + a_{22}y + a_{23}z + a_{24} \\
z &= a_{31}x + a_{32}y + a_{33}z + a_{34} \\
1 &= a_{41}x + a_{42}y + a_{43}z + a_{44}
\end{align*}
\]

To be useful in kinematic applications these equations may be put in homogeneous form by adding a fourth variable (dimension) \( w \). In matrix form these equations become:

\[
\begin{bmatrix}
x \\ y \\ z \\ w
\end{bmatrix} = 
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44}
\end{bmatrix}
\begin{bmatrix}
x_1 \\ y_1 \\ z_1 \\ w_1
\end{bmatrix}
\]

It can be shown that the submatrix

\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}
\end{bmatrix}
\]

describes a rotation about a fixed axis in space passing through the origin of the coordinate system, as might be obtained by direct substitution into equation (3). The determinant of the rotation matrix is always equal to unity.

If in equation (5) \( w \) is assumed equal to unity, the transformation is said to describe the motion of a three-dimensional set of points contained within a rigid body relative to the hyperplane \( w = 1 \).

General rigid body motion is therefore described by a \( 4 \times 4 \) numerical matrix with a \( 3 \times 3 \) submatrix containing all the information concerning rotation of the rigid body in space. The coefficients in the column matrix \( a_{14}, a_{24}, a_{34} \) are the translation components. Equation (5) then becomes:

\[
\begin{bmatrix}
x \\ y \\ z \\ 1
\end{bmatrix} = 
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\ y_1 \\ z_1 \\ 1
\end{bmatrix}
\]

**Displacement of a Plane in Three-Dimensional Space**

Consider a general rigid body motion in space in which the coordinates of three arbitrary points forming a plane in the rigid body are given in positions 1 and 2.

\[
\begin{align*}
A_1 &= A_1(1, 1, 0) \\
A_2 &= A_2(5, 2, 0) \\
B_1 &= B_1(1, 4, 0) \\
B_2 &= B_2(8, 2, 0) \\
C_1 &= C_1(1, 1, -1) \\
C_2 &= C_2(5, 2, 1)
\end{align*}
\]

Assume that the coordinates can be related by a \( 3 \times 3 \) matrix transformation of the form:

\[
\begin{bmatrix}
x_1 \\ y_1 \\ z_1
\end{bmatrix} = 
\begin{bmatrix}
x_2 \\ y_2 \\ z_2
\end{bmatrix}
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}
\end{bmatrix}
\]

This would intuitively seem correct and the nine unknowns \( a_{ij} \) could be found by matrix inversion from the nine known coordinates in each position as follows:

\[
\begin{bmatrix}
5 & 8 & 5 \\ 2 & 2 & 2 \\ 0 & 0 & 1
\end{bmatrix} = 
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 \\ 1 & 4 & 1 \\ 0 & 0 & -1
\end{bmatrix}
\]

\[
[a_{ij}] = 
\begin{bmatrix}
5 & 8 & 5 \\ 2 & 2 & 2 \\ 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 \\ 1 & 4 & 1 \\ 0 & 0 & -1
\end{bmatrix}^{-1} = 
\begin{bmatrix}
4 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1
\end{bmatrix}
\]

When substituted into equation (7), it can be shown that the \( 3 \times 3 \) matrix will transform only those points which lie in the plane of the three points \( ABC \) from position 1 to the proper position 2. Contrary to what might have been expected, other points in the rigid body containing \( ABC \), but which do not lie in plane \( ABC \), will not be displaced to the proper point in position 2. A \( 3 \times 3 \) matrix will be useful in describing the displacement of a two-dimensional plane surface in three-dimensional space. A \( 4 \times 4 \) matrix is required for the displacement of a three-dimensional rigid body in the particular four-dimensional space with \( w = 1 \).

The matrix of equation (8) may be tested for simple rotation about a fixed axis by evaluating its determinant. In this case the determinant is equal to \(+2\), which indicates that the matrix does not represent three-dimensional rigid body displacement.

**Spherical Rigid Body Displacement**

The displacement matrix for spherical rigid body motion can be evaluated in a manner similar to that used in the previous section. Any four non-coplanar points on the rigid body with their coordinates specified in positions 1 and 2 may be used for evaluation of the displacement matrix.

In spherical motion it is convenient to include the fixed origin among the four non-coplanar points. Since spherical motion has only rotational displacement components if the fixed center of the sphere has the same coordinates as the origin \((0, 0, 0)\), the spherical displacement matrix will always have the form:

\[
[D_{rs}] = 
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & 1
\end{bmatrix}
\]

Consider rigid body spherical displacement of three points plus the origin as follows, where primed coordinates refer to the position after displacement.

\[
\begin{align*}
P_1 &= P_1(x_1, y_1, z_1, 1) \\
P_2 &= P_2(x_2, y_2, z_2, 1) \\
P_3 &= P_3(x_3, y_3, z_3, 1) \\
P_4 &= P_4(0, 0, 0, 1)
\end{align*}
\]

Then

\[
\begin{bmatrix}
x_1' \\ x_2' \\ x_3' \\ z_1' \\ z_2' \\ z_3'
\end{bmatrix} = [D_{rs}]
\begin{bmatrix}
x_1 \\ x_2 \\ x_3 \\ z_1 \\ z_2 \\ z_3
\end{bmatrix}
\]

from which

\[
[D_{rs}] = 
\begin{bmatrix}
x_1' & x_2' & x_3' & 0 \\ y_1' & y_2' & y_3' & 0 \\ z_1' & z_2' & z_3' & 0 \\ 1 & 1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2 \\ x_3 \\ z_1 \\ z_2 \\ z_3
\end{bmatrix}^{-1}
\]

As in the case of plane motion it is often convenient to form the displacement matrix for spherical motion by considering the motion of any two points (plus the origin). In this case, assuming the coordinates of two points \( P_1 \) and \( P_2 \) are to be used, the coordinates of a third point may be derived by rotating \( P_1P_2 \) 90 deg about an axis \( OP_1 \).

From the rotation matrix given in equation (3) with \( u_z = z_1, u_y = y_1, u_x = 0, \) and \( \phi = 90 \) deg, the coordinates of two third points \((x_2', y_2', z_2')\) and \((x_3', y_3', z_3')\) are calculated, respectively. Therefore, we have
5 General Method of Synthesis for Spherical Linkages

As in plane linkage synthesis, the general problem in spherical linkage synthesis involves the determination of the first position of possible moving axes on the moving rigid body which pass through a sequence of positions which pass through a circular arc on the surface of a sphere. Two spherical links, as shown in Fig. 1, connected between each of a pair of such points (spherical circle points) and corresponding fixed points (spherical center points) which lie on axes passing through the origin and the center of the circular arcs, would guide the moving body through a series of prescribed positions.

Spherical linkages for rigid body guidance are synthesized by the iterative solution of a set of simultaneous nonlinear algebraic design equations. The iterations are carried out with a digital computer using the Newton-Raphson procedure.

The design equations are developed as follows:

1 Assuming the rigid body motion is known to be spherical, the coordinates of the center (O_x, O_y, O_z) and the radius of the sphere are calculated from the following equations:

\[
\begin{bmatrix}
(x^2 + y^2 + z^2) \\
(x_1^2 + y_1^2 + z_1^2) \\
(x_2^2 + y_2^2 + z_2^2) \\
(x_3^2 + y_3^2 + z_3^2)
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix}
= 0
\]

which may be expanded in the form

\[
x^2 + y^2 + z^2 + Ax + By + Cz + D = 0
\]

The subscripted terms in equation (11) refer to the four positions of any given point. Reducing equation (12) to the form

\[
(x - O_x)^2 + (y - O_y)^2 + (z - O_z)^2 = R^2
\]

we find the radius \( R \) and the coordinates of the center (O_x, O_y, O_z).

2 The spherical equation (13) is then normalized to the equation of a unit sphere with its center at the origin (0, 0, 0) by dividing by \( R^2 \) and shifting the center by translation of the coordinate system through

\[
\left( \begin{array}{c}
\frac{O_x}{R} \\
\frac{O_y}{R} \\
\frac{O_z}{R}
\end{array} \right)
\]

The coordinates of the moving points are then converted to the normal coordinate system and the synthesis carried out on the surface of the unit sphere.

3 The displacement matrices [Dn] are calculated in normal form from any of equations (3), (9), or (10).

4 Design equations are based upon constraint equations which ensure constant length of guiding links and the equations of the unit sphere. The coordinates of a spherical center point and circle point are designated \( A_0 = A_0(x_0, y_0, z_0) \) and \( A_1 = A_1(x_1, y_1, z_1) \), respectively.

The constraint equations become

\[
\begin{align*}
(x_0 - x_1)^2 + (y_0 - y_1)^2 + (z_0 - z_1)^2 &= 0 \\
(x_0^2 + y_0^2 + z_0^2) &= 1
\end{align*}
\]

6 Synthesis for Three-Position Guidance by Desk Calculator

Assume a typical problem in which three spherical positions of a rigid body are given along with the first position of two points on arbitrary moving pivot axes. The locations of the corresponding fixed pivots are desired. The positions of points on the moving body might also be specified in terms of initial position and rotation matrices given by equation (3).

In order for the moving points to be spherical circle points, they must lie in a circular arc on the surface of the sphere. It will always be possible to define the plane \( \pi \) containing the three positions of a circle point from the equation

\[
\begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix}
= 0
\]
A plane \( \pi' \), parallel to \( \pi \) and tangent to the sphere, will locate the fixed pivot \( A_0 = A_0(x_0, y_0, z_0) \) at the point of tangency. This results in the pair of equations

\[
\begin{bmatrix}
  y_0 & z_0 & 1
\end{bmatrix}
\begin{bmatrix}
  x_0 & x_1 & x_2 & x_3
\end{bmatrix}
= \begin{bmatrix}
  x_0 & y_1 & z_1 & 1
\end{bmatrix}
\begin{bmatrix}
  x_0 & x_0 & x_0 & x_0
\end{bmatrix}
\]

The design equations become,

\[
Ay_0 - Bz_0 = 0
\]
\[
A_x = -Cz_0 = 0
\]
\[
x_i^2 + y_i^2 + z_i^2 = 1
\]

where

\[
A = \begin{bmatrix}
  y_1 & z_1 & 1
\end{bmatrix}
\begin{bmatrix}
  x_0 & x_1 & x_2 & x_3
\end{bmatrix}
\]
\[
B = -x_1 2i 1
\]
\[
C = \begin{bmatrix}
  x_1 & y_1 & 1
\end{bmatrix}
\begin{bmatrix}
  x_0 & x_0 & x_0 & x_0
\end{bmatrix}
\]

which leads to

\[
x_0 = \pm \frac{A}{\sqrt{A^2 + B^2 + C^2}}
\]
\[
y_0 = \pm \frac{B}{\sqrt{A^2 + B^2 + C^2}}
\]
\[
z_0 = \pm \frac{C}{\sqrt{A^2 + B^2 + C^2}}
\]

The plus or minus sign determines one of two possible locations for \( A_0 = A_0(x_0, y_0, z_0) \) corresponding to the two intersections of the fixed axis with the surface of the sphere.

### 7 Synthesis for Four-Position Guidance

The relative motion occurring at the human hip joint closely approximates spherical motion. The following example involves a study of the feasibility of constructing an external mechanical analog of human hip motion which might serve as an orthopaedic brace, i.e., a linkage which would guide the femur through a series of positions relative to the pelvis.

Using the pelvis as a reference, the relative motion between femur and pelvis was measured giving four relative positions of two selected points, \( P_i \) and \( P_i' \), moving with the femur (Fig. 2). These coordinates were used to calculate displacement matrices \([D_1],[D_3]\), and \([D_4]\) which may be used to determine the displacement of any assumed point moving with the femur.

The normalized displacement data are:

\[
P_1 = P_1(0.105040, 0.482820, 0.869397)
\]
\[
P_2 = P_2(0.090725, 0.541283, 0.835931)
\]
\[
P_3 = P_3(0.104155, 0.620000, 0.777656)
\]
\[
P_4 = P_4(0.096772, 0.725995, 0.681173)
\]
\[
P_1' = P_1'(-0.404640, -0.676760, 0.571057)
\]
\[
P_2' = P_2'(-0.133748, -0.751642, 0.645868)
\]
\[
P_3' = P_3'(0.161113, -0.702067, 0.693846)
\]
\[
P_4' = P_4'(0.400762, -0.564306, 0.721769)
\]

Using equations (14) and referring to Table 1, we note that for any assumed coordinate \( z_0 \) we may solve for \( y_0, x_0, y_1, \) and \( z_1 \). After the initial solution, we may increment either \( x_0, y_0, \) or \( z_0 \) and continue a series of solutions which are plotted as spherical circle and center-point curves. The curves of Fig. 2 are plotted from the computed numerical values and indicate possible spherical linkage arrangements which could form the basis of the proposed orthopaedic brace for a human hip joint. The curves are shown as projected on the \( xy \)-plane. Once a linkage has been selected, actual numerical point coordinates are available from the computer printout.

It is interesting to note that spherical circle-point and center-point curves generally form closed loops on the surface of the sphere. Secondly, since either a circle point or center point defines an axis in the sphere, they exist as dual points or point pairs on opposite sides of the sphere, as indicated previously in equations (18).

### 8 Synthesis of Spherical Linkage Function Generators

As mentioned in the introduction, a spherical linkage function generator is synthesized by kinematic inversion about the first position of the output link.

With reference to Fig. 3, the motion \( \theta \) of input link 2 is proportional to the independent variable \( x_i \), and the motion \( \phi \) of output
The output axis at \( v_x = y = 0, u_x = z = 0 \), is arbitrarily as-

Substitution into equation (3) leads to

\[
[D_{in}]y_a = \begin{bmatrix} A_{in} & B_{in} & C_{in} & 0 \\ D_{in} & E_{in} & F_{in} & 0 \\ G_{in} & H_{in} & I_{in} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

where

\[
A_{in} = \cos^2 \alpha + \sin^2 \alpha \cos \phi_{in}
\]

\[
B_{in} = \sin \alpha \cos \alpha \cos \theta_{in}(1 - \cos \phi_{in})
\]

\[
C_{in} = -\sin \alpha \cos \alpha \sin \theta_{in}(1 - \cos \phi_{in})
\]

\[
D_{in} = \sin \alpha \cos \phi_{in}
\]

\[
E_{in} = \sin^2 \alpha \cos \theta_{in} + \cos^2 \alpha \cos \phi_{in} \cos \theta_{in}
\]

\[
F_{in} = -\sin \alpha \cos \theta_{in} - \cos \alpha \sin \phi_{in}
\]

\[
G_{in} = \sin \alpha \sin \phi_{in}
\]

\[
H_{in} = \sin \phi_{in} \cos \theta_{in} - \cos \alpha \cos \theta_{in} \sin \phi_{in}
\]

\[
I_{in} = \cos \theta_{in} \cos \phi_{in} + \cos \alpha \sin \theta_{in} \sin \phi_{in}
\]

With displacement matrices \([D_{in}]y_a\) available in numerical form, equations (14) can be used to locate relative circle points \( B_i = B_i(z_i, y_i, z_i) \) and relative center points \( C_i = C_i(x_i, y_i, z_i) \). Since the spherical function generator is a special case of a spherical rigid body guidance problem, a maximum of five precision points is possible for an arbitrarily assumed value of \( \alpha \), as indicated in Table 1. A maximum of six precision points is theoretically possible if \( \alpha \) is considered as a variable in the design equations.

9 Synthesis of Spherical Linkages for Path Generation

As shown in a previous paper [12] dealing with plane linkages, it is convenient when dealing with path generation mechanisms to have the displacement matrix in analytical form in terms of the displacement of a specified point in a rigid body and an angular displacement of the rigid body about an axis passing through the specified point. In the case of spherical linkages for path generation, the path is generated by a point attached to the coupler and the form of the displacement matrix must insure spherical coupler motion.

With reference to Fig. 4, we specify a spherical motion of a rigid body which will form the coupler in the final linkage. A point \( P \) on the body is displaced from \( F_i = P_i(z_i, y_i, z_i) \) to \( P_e = P_e(z_e, y_e, z_e) \).

The spherical displacement of the coupler is resolved into two components: A great circle rotation \( \alpha_{in} \), plus a rotation \( \beta_{in} \) about the axis \( OP_{in} \). These two components are shown in Fig. 4. If the great circle was the equator of the earth, the rotation \( \alpha_{in} \) would be the spherical displacement about the north-south pole axis and the rotation \( \beta_{in} \) the final inclination of a line originally lying on the equator.

The great circle rotation corresponds to a spherical angle given by

\[
\cos \alpha_{in} = x_in x_n + y_in y_n + z_in z_n
\]

The rotation axis for \( \alpha_{in} \) is perpendicular to \( P_i P_e \) and passes through the point \( O_{in} \). The axis \( OO_{in} \) must lie on the vector cross product \( V_{in} \) where

\[
V_{in} = OP_i \times OP_{in}
\]

Therefore

\[
O_{in} = O_i(x_in, y_in, z_in),
\]

where

\[
x_in = -\frac{Q_{in}}{+\sqrt{Q_{in}^2 + R_{in}^2 + S_{in}^2}}
\]

\[
y_in = -\frac{R_{in}}{+\sqrt{Q_{in}^2 + R_{in}^2 + S_{in}^2}}
\]

\[
z_in = \frac{S_{in}}{+\sqrt{Q_{in}^2 + R_{in}^2 + S_{in}^2}}
\]

and

\[
Q_{in} = y_in x_n - y_n x_in, \quad R_{in} = x_in z_n - z_in x_n, \quad S_{in} = x_n y_n - y_n z_n.
\]

In deriving the displacement matrix, we account for the rotation \( \beta_{in} \) first, followed by the great circle rotation \( \alpha_{in} \). The displacement matrix is found from the matrix product

\[
[D_{in}] = [D_{in}]_{\beta_{in} \times \alpha_{in}} [D_{in}]_{\alpha_{in} \times \beta_{in}}
\]

The displacement matrices from equation (24) will involve the coordinates of the specified precision points \( (x_in, y_in, z_in) \) and the unspecified rotation angles \( \beta_{in} \).

Fig. 5 shows a typical spherical linkage path generation mechanism in a position corresponding to the first path precision point. Table 2 lists the specified and unspecified parameters as a function of the number of specified path precision points. Note that with up to five precision points, the guiding cranks are independently determined if the angles \( \beta_i \) are all specified. With more than five precision points the two cranks are always mutually dependent upon the choice of specified parameters. The iterative solution of the design equations results in the possibility of a maximum of nine path precision points.
Example Problem. Design of a Spherical Linkage Path Generation Mechanism with Four Path Precision Points in Space.

Five coordinates \( p_0, r_0, p_1, r_1, s_1 \) are specified. Ten variables \( q_0, q_i, s_0, t_0, u_0, t_1, u_1, \beta_0, \beta_1, \beta_1 \) are calculated.

Problem: Design a spherical linkage to guide a point \( P \) through four points in space. (Precision points as given in example of reference [4].)

\[
P_1 = P_1(1.00, 1.00, 2.00) \\
P_2 = P_2(2.50, 1.75, 2.50) \\
P_3 = P_3(4.00, 2.00, 2.00) \\
P_4 = P_4(4.50, 1.00, 1.00)
\]

We normalize the coordinates by first using equation (11) expanded into the form of equation (12) to obtain

\[
x^2 + y^2 + z^2 - 4.1875x - 5.4375y + 1.59375z + 0.4375 = 0
\]

which reduces to

\[
(x - 2.093750)^2 + (y - 2.718750)^2 + (z + 0.796875)^2 = (3.460190)^2
\]

The coordinates of the sphere center and radius are

\[
x_0 = 2.093750, \quad y_0 = 2.718750, \quad z_0 = -0.796875 \\
R = 3.460190
\]

The unit sphere coordinates are

\[
X = \frac{x}{3.460190} - 0.605096 \\
Y = \frac{y}{3.460190} - 0.785722 \\
Z = \frac{z}{3.460190} + 0.230297
\]

where

\[
X^2 + Y^2 + Z^2 = 1.
\]

The coordinates of point \( P \) on the unit sphere are designated as \( P' \).

\[
P'_1 = P'_1(-0.316095, -0.496721, 0.808300) \\
P'_2 = P'_2(0.171407, -0.279070, 0.925801) \\
P'_3 = P'_3(0.559000, -0.207720, 0.808300) \\
P'_4 = P'_4(0.695410, -0.003725, 0.808300)
\]

Specify \( p_0, r_0, p_1, r_1, \) and \( s_1 \) arbitrarily

\[
p_0 = 0.100000, \quad p_1 = -0.600000 \\
r_0 = 0.200000, \quad r_1 = 0.400000
\]

From \( p_0^2 + q_0^2 + r_1 = 1 \) and \( p_1^2 + q_1^2 + r_1^2 = 1 \), we obtain

\[
q_0 = -0.974679, \quad q_1 = 0.6092820
\]

which completely specifies one crank \( A_1'A_1' \) on the unit sphere in the first position.

From the condition that the distance \( A_1'P_1' \) is constant, we may calculate

\[
A_1' = A_1'(-0.600000, -0.692820, + 0.400000) \\
A_1' = A_1'(-0.127438, -0.385503, 0.761718) \\
A_1' = A_1'(0.124256, -0.529236, 0.839057) \\
A_1' = A_1'(0.251528, -0.502039, 0.816268)
\]

From the known positions of two points on the coupler we may now calculate the displacement matrices \([D_{01}], [D_{02}], [D_{04}]\) and proceed as outlined with equations (14) in the rigid body guidance section for the determination of the second crank.

A second method for finding the second crank, which may be used for a desk calculator solution, is based upon the fact that the four positions of a spherical circle point lie in a common plane. Therefore, we may write

\[
s_0, t_0, u_0, t_1, u_1, 1 = 0
\]

Choose \( s_0 = 0 \) and, using the displacement matrices, replace all elements of the determinant by their equivalents in terms of \( t_i \) and \( u_i \). Expansion of the determinant results in a third-order algebraic equation

\[
0.00926284t^4 + 0.00372549t^2 + 0.026118u^2 + 0.025851u^2 = 0
\]

Since \( t_i^2 = 1 - u_i^2 \), this reduces to a cubic in \( u_i^2 \)

\[
(u_i^2)^3 - 0.621560(u_i^2)^2 + 0.131498(u_i^2) - 0.005352 = 0
\]

which gives a single real solution

\[
(u_i^2) = 0.652720, \quad t_i = \pm 0.22988
\]

Transactions of the ASME
Choosing the negative value of \( \theta \) such that the moving pivot lies in the \(-y\) hemisphere, we see that only the positive value of \( \theta \) is consistent with the third-order equation above.

Once the first position \( B_1' \) is specified, \( B_2', B_3', \) and \( B_4' \) are calculated using the displacement matrices. The fixed pivot is located from any three of the positions of \( B' \) using the method of section 6, giving:

\[
B_1' = B_1'(0.000000, -0.973283, 0.229608)
\]

\[
B_2' = B_2'(0.249848, -0.953447, -0.168485)
\]

The normalized design coordinates may be transformed to the original coordinate system using equations (25).

A digital computer solution based upon equations (14) resulted in the center-point and circle-point curves shown in Fig. 6, which are plotted as projected on both the \(+y\) and \(-y\) hemispheres in order to display the closed loop and dual point characteristics of the curves. Note that in this case a single closed curve is continuous on both hemispheres, while in the previous four-position guidance example, separate closed curves resulted in each hemisphere. In either situation, a plot on a single hemisphere would display all computed solutions due to the dual nature of the points on a common diameter of the sphere.

10 Application of the Displacement Matrix to Position Analysis of Spherical Mechanisms

Input-Output Analysis of a Spherical Function Generator
The relative position of output crank to input crank may be specified in terms of the relative displacement matrix \([D_{ii}']\), given in equation (20) as a function of the known input angle \( \theta \) and unknown output angle \( \phi \). Point \( B \) on the coupler is constrained to move in a circular arc about point \( C \) on the output crank; hence, we may impose the constraint equation

\[
(x_a - x_c)^2 + (y_a - y_c)^2 + (z_a - z_c)^2 = (x_b - x_c)^2 + (y_b - y_c)^2 + (z_b - z_c)^2
\]

where, in this case, \( B_2 = B_2(x_a, y_a, z_a) \) in the \( n \)th position of point \( B \) relative to \( C \), and \( B_1 = B_1(x_1, y_1, z_1) \) is the first position.

Solution of equation (27) with \( (x_a, y_a, z_a) \) calculated using equation (20) will give the output angle \( \phi \) for any specified input angle \( \theta \). The previous position provides all initial guesses for the iterative solution in the next position.

Position Analysis in Spherical Rigid Body Guidance or Path Generation Linkages
Position analysis in rigid body guidance linkages is carried out in two steps: (a) an input-output analysis of the crank motions as indicated in the foregoing, from which the positions of moving pivots can be specified, and (b) calculation of the coupler displacement matrix \([D_{ij}]\) which allows the calculation of the \( n \)th position of any point in the coupler.

11 Conclusion
The methods outlined in the present paper, when used with either desk calculator or digital computer, have proved to be very practical for the synthesis of spherical linkages. The method may be extended to bevel gear-spherical linkage combinations in a manner similar to that demonstrated in a previous paper on plane linkage synthesis [12].

Computer programs written in FORTRAN IV language for the synthesis of rigid body guidance, function generation, and path generation mechanisms with up to five precision points are available from the authors upon request. Programs for problems with higher numbers of precision points are being developed.

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