KINEMATIC STRUCTURE OF MECHANISMS

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ABSTRACT: An elementary survey is presented of basic problems and
techniques in the kinematic structure of mechanisms. The discussion
is limited to the application of graph theory and includes a dis-
cussion of current research.

1. INTRODUCTION

The structure of many systems, be they electrical circuits
or chemical compounds, is basic to an understanding of their
function. This has been realized for a long time in diverse
areas, ranging from molecular biology to social systems. It is
relatively recent, however (1965), that this point of view has
been applied to mechanisms and most mechanical systems.

We shall be concerned here with the kinematic structure of
mechanisms. An understanding of this structure may be expected
to lead to (a) a systematic development of the statics, kinematics
and dynamics of rigid-body mechanisms and mechanical systems, (b)
to a logical system of structural classification of mechanisms
and mechanical systems and (c) to a foundation for the analysis of
more general mechanical systems, involving a wider variety of
components, and including both rigid-body- and elastic displace-
ments.

To begin with, however, in order to define our subject, let
us review the definition of a mechanism. This involves a sequence
of definitions as follows.

Definition 1a: Restricted relative motion: relative motion be-
tween two rigid bodies in Euclidean space having a degree of
freedom less than 6 and greater than 0.

Definition 1b: Kinematic element: part of a rigid body having
a form which permits connection (pairing) to another kinematic
element on another rigid body in such a way that a restricted
relative motion between the bodies is permitted.

Definition 1c: Two kinematic elements paired as in Definition 1b,
are said to constitute a kinematic pair or joint.

A list of kinematic pairs is shown in Table 1.
I. Kinematic elements with uncoupled rotational and translational freedoms, as indicated.

<table>
<thead>
<tr>
<th>Sphere between parallel planes.</th>
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<tbody>
<tr>
<td>(3-2)</td>
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<tr>
<td>Ball in cylinder</td>
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<tr>
<td>Cylinder between parallel planes (2-2)</td>
</tr>
<tr>
<td>(1-3)</td>
</tr>
<tr>
<td>Ball joint</td>
</tr>
<tr>
<td>Sphere in slotted cylinder (2-1)</td>
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<tr>
<td>Plane joint</td>
</tr>
<tr>
<td>(0-2)</td>
</tr>
<tr>
<td>(1-2)</td>
</tr>
<tr>
<td>Slotted Sphere</td>
</tr>
<tr>
<td>Torus</td>
</tr>
<tr>
<td>Cylindrical Joint</td>
</tr>
<tr>
<td>Roller in Slot</td>
</tr>
<tr>
<td>(0-1)</td>
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<tr>
<td>(1-1)</td>
</tr>
<tr>
<td>Turning Pair</td>
</tr>
<tr>
<td>(1-0) Prismatic Pair</td>
</tr>
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</table>

II. Coupled Kinematic elements

<table>
<thead>
<tr>
<th>f=1 Helical Pair</th>
</tr>
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<tbody>
<tr>
<td>f=1 Circular Slider in circ. slot.</td>
</tr>
<tr>
<td>Gear Pair, f=2</td>
</tr>
<tr>
<td>Cam pair, f=2</td>
</tr>
<tr>
<td>Noncircular gear pair, f=2</td>
</tr>
<tr>
<td>f=1 Constant-breadth cam</td>
</tr>
<tr>
<td>f=2 Constant-breadth cam</td>
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**TABLE 1**
Definition 1d: Link: a rigid body with at least two kinematic elements.
Definition 1e: Closed kinematic chain: a set of links, each of which is pair-connected to at least two other links.
Definition 1f: Mechanism: a closed kinematic chain, one link of which (called the frame) is considered fixed, a restricted relative motion between the links being possible.

Fig. 1 shows some typical basic mechanisms: a slider-crank mechanism (Fig. 1a), a four-bar linkage (Fig. 1b) and a gear train (Fig. 1c).

The kinematic structure of mechanisms is concerned with those properties of mechanisms, which are determined solely by the kinematic pairing (and which are, therefore, independent of metric properties). Degree-of-freedom analysis belongs in this category. Mechanisms for which the degree of freedom depends on metric properties and these can be important are excluded in our analysis. By their very nature mechanisms have a positive degree of freedom. The case of zero or negative freedom belongs in the area of structures (on this see, for instance: J.E. Benveniste: "Study of mechanisms with applications to frames", J.Eng. Mech. Div. ASCE, 91, EM1, Feb. 1965, pp. 19-50).

In what follows we shall be concerned solely with techniques involving the graph representation of kinematic structure. Hence, the discussion of prior work can be correspondingly brief.

2. Previous work

This refers primarily to the degree-of-freedom analysis of linkages and to a lesser extent to the structural classification of mechanisms.

(a) Degree-of-freedom analysis

The "link-joint" degree-of-freedom equations were initiated by Chebyshev (6) (1869), Sylvester (1874)* and Gruebler (22) (1883-85) and have been the subject of numerous investigations.

For example, for plane, pin-connected linkages with \( \ell \) links and \( j \) joints, Gruebler showed that the degree of freedom, \( F \), is generally given by:

\[
F = 3\ell - 2j - 3
\]  \hspace{1cm} (1)

Gruebler also gave equations relating the number of links to the number of independent mechanical circuits, and developed the subject systematically.

The "mobility" concept in the degree-of-freedom analysis of mechanisms is due to P.L. Somov (29) (1887). For constrained (i.e. \( F=1 \)), pin-connected, plane linkages, Somov derived the equation:

\[
\ell - L (\lambda - 1) = 2
\]  \hspace{1cm} (2)

where \( L \) denotes the number of independent mechanical circuits. The equation remains valid for three-dimensional linkages, provided the degree of freedom of each joint is unity. The number \( \lambda \), called the mobility number, is defined as the degree of freedom of the individual, unconstrained members of the kinematic chain and is assumed constant for all members. Typical values for \( \lambda \) would be 3 for plane motion or motion on a surface, and 6 for general, three-dimensional motion.

The combination of the mobility concept with the "link-joint" type of freedom equation has led various investigators (Artobolevskii, Dobrovolskii, Kutzbach, Rudolf Mueller etc.) towards a general form of the degree-of-freedom equation for both plane and three-dimensional mechanisms (see Artobolevskii (1)) and Federhofer (15)):

\[
F = \lambda (\ell - j - 1) + \sum_{i} j_i.
\]

where \( \lambda \) denotes the mobility number (assumed constant) for each link of the mechanism and \( j_i \) denotes the number of joints the degree of freedom associated with which is \( i \). It is understood that a joint connecting \( n \) links counts as \( (n-1) \) joints.

The derivation of (3) is outlined in Appendix 1.

(b) Structural classification of mechanisms.

In his 1883-85 investigations, Gruebler attempted the enumeration of the constrained (\( F=1 \)), pin-connected, eight-bar kinematic chains and listed 12. Later on H. Alt is reported to have discovered 4 more, but his results were not published. We now know that there are 16 eight-bar kinematic chains and 71 mechanisms derivable from these (see, for instance: K. Hain: "Die Analyse und Synthese der achtgliedrigen Gelenkgetriebe, VDI-Berichte 5, 1955, 2-14). The first description of these, as far as is known to us, can be found in A.W. Klein's "Kinematics of Machinery" (McGraw-Hill, 1917), including also a statement that there are 228 constrained 10-bar kinematic chains.

*In this connection it may be worth noting the essentially prophetic remarks on this subject by J.J. Sylvester ("On recent discoveries in the mechanical conversion of motion", Collected Papers, Vol. 3, 1870-73): "Thus we see that a purely tactical theory of colligation underlies the subject of linkages ... it takes no account of magnitude or position: geometrical lines are used, but have no more real bearing on the matter than those employed in genealogical tables have in explaining the laws of procreation". If only we could write up our work as colorfully today!
Fig. 1: Basic mechanisms: (a) Slider-crank mechanism (b) Four-bar linkage (c) Epicyclic gear train.
L.W. Assur (2) (1912-16) showed how to build up complex linkages from simple linkages by means of the addition of link dyads, leading to "families" of mechanisms of quite general character.

R. Franke (16) (1948-50) developed a "condensed notation" for the structural classification of mechanisms (11). It can be shown that in mathematical terms this is equivalent to the enumeration of the contracted graphs discussed by L.S. Woo (31).

These investigations were conducted prior to the application of graph theory to mechanisms. Questions such as when two mechanisms were the "same" or "different", were not always defined, while a good deal of intuition was ably used in many cases.

The topological nature of the subject was mentioned by B. Paul (26) (1960), who introduced Euler's formula for polyhedra to show that the number of fundamental circuits, \( L \), for mechanisms represented by planar graphs is given by:

\[
L = 1 + j - \ell
\]  

(4)

The introduction of graph theory to mechanisms was given in 1965 (F.R.E. Crossley (7) and F. Freudenstein and L. Dobrjanskyj (17)) and we turn to this next.

3. APPLICATION OF GRAPH THEORY TO THE KINEMATIC STRUCTURE OF MECHANISMS

a. Definition

The graph of a mechanism with binary joints (each joint connecting two links) can be defined as a linear graph, in which links correspond to vertices, joints correspond to edges, and the kinematic pairing of links corresponds to the edge-connection of vertices; edges are labeled according to joint type and the vertex representing the fixed link is labeled accordingly.

With this definition, the number of independent mechanical circuits of the mechanism corresponds to the number of fundamental circuits of the graph* and the structural equivalence of mechanisms corresponds to the isomorphism of their graphs.

\*Here we define a fundamental circuit as consisting of a chord and its unique, associated tree path. When the graph is planar, we have the simpler, equivalent result given by Euler's formula for polyhedra.
For example, the graph of the mechanism of Fig. 1c is shown in Fig. 2.

Fig. 2: Graph of gear train of Fig. 1c; heavy lines denote geared edges, light lines denote turning edges; vertex 1 identified as representing fixed link.

It is possible to devise other definitions of the graph of a mechanism, but the above appears to be the most useful.

When the joints are not binary, the definition needs modification. Although it is possible to derive any mechanism with non-binary joints (these are called multiple joints, because they connect more than two links) by specialization of a mechanism having only binary joints, such a procedure is not always convenient. If the given definition is not restricted to mechanisms with binary joints, the graph of such mechanisms will have independent circuits, which involve only the links connected at the multiple joint. Such a circuit does not possess the desirable property of corresponding to a mechanical circuit with which we may associate a (loop-closure) displacement equation. One method of avoiding this difficulty is to resort to multiple labeling of the edges: one label designating the type of joint, and the other designating the principal geometrical element (axis, point or plane) associated with the joint. If all the edges of a fundamental circuit are associated with the same principal geometrical element, the circuit represents simply a multiple-joint connection.

Such a definition can be useful in the computer-aided kinematic analysis of mechanisms, which ideally proceeds automatically, given only the kinematic structure and the principal dimensions of the mechanism. It is possible to define the graph of mechanisms with multiple joints in different ways and the above represents one such possibility.

(b) The basic degree-of-freedom equations
These are most simply and unambiguously obtained from the graph representation of kinematic structure. The most important of these are:

\[ F = \lambda (l - j - l) + \sum_{i} r \cdot l_i \]  \hspace{1cm} (5a)

\[ F = \lambda (l - j - l) + \sum_{i} f_i \]  \hspace{1cm} (5b)

\[ F = \sum_{i} f_i - \lambda L \]  \hspace{1cm} (5c)
\[ L = 1 + j - \ell \] (5d)

\[ \sum_i i \ell_i = \sum_k k \ell_k = 2j \] (5e)

where \( \ell \) denotes the degree of freedom associated with the \( i \)th joint, \( \ell_i \) the number of links having \( i \) elements, and \( L \), the number of independent mechanical circuits having \( K \) joints (edges), now also including the peripheral loop. Eqs. (5b, c) can be derived from eqs. (5a, d) so that only 2 of the first 4 equations are independent.

Many other equations and inequalities can be derived, e.g.:

\[ \ell_2 \geq (3\ell - 2j) \] (6)

It is possible also to investigate the statical determinacy of mechanisms in the absence of friction, assuming (a) the kinematic pairs are of the uncoupled type (i.e. the degrees of freedom in rotation and translation are independent) (b) the only external forces and torques are applied at the elements connected to ground. Let \( F_T \) denotes the degrees of freedom associated with the joints connected to ground and \( P \) denotes the number of prescribed external force-and torque components acting along axes associated with the degrees of freedom of the terminal joints (those connected to ground); then for statical determinacy,

\[ P = F_T - F \] (7)

The derivation is given in Appendix 2.

(c) Structural classification of mechanisms

The problem is the enumeration and construction of the kinematic structure of mechanisms having a given degree of freedom, specified types of kinematic elements and (possibly) a given number of links.

In terms of graph representation, the problem can be formulated as a graph enumeration problem, generally as a graph-coloring problem.

In the case of plane bar linkages, it is convenient to enumerate the kinematic chains, since the mechanisms are obtained by kinematic inversion. For example, the enumeration of constrained eight-bar linkages mentioned in Section 2(b) is equivalent to the enumeration of linear graphs having 8 vertices and 10 edges, the degree of each vertex being between 2 and 4, and no loop having less than 4 edges. In this case all the edges represent turning pairs and hence need not be labeled according to type. Thus, the problem of graph enumeration posed by even relatively simple classes of mechanisms, is not necessarily an elementary one.
The case of eight-link kinematic chains has been discussed by Crossley (7), Freudenstein and Dobrjanskyyj (17), who have confirmed the 16 kinematic chains ascribed to H. Alt, as mentioned earlier.

The enumeration of constrained, plane ten-bar linkages has been considered by Crossley (8, 9) and Woo (31). The latter has shown, using Burnside's Theorem and the "contraction map" of a graph, that there are 230 distinct ten-link constrained kinematic chains. The contraction map is defined as the subgraph of the graph of a kinematic chain (this differs from the graph of a mechanism only in that the vertex identification of the fixed link is omitted) in which all vertices of degree 2 have been deleted. The enumeration is formulated in terms of two subsidiary problems: the enumeration of the contracted graphs and the coloring of the contracted graphs according to the number of vertices to be added to each edge. Burnside's Lemma (see Appendix 3) is useful in this connection.

Unpublished studies of C.W. McLarnan (24) have considered the enumeration of constrained 12-bar, pin-connected linkages (kinematic chains) by means of algorithms, a recursive approach also described by L.S. Woo (31). The number of distinct kinematic chains increases very rapidly with the number of links, so that the effort involved can only be described as awesome.

Polya's theorem has also been applied to the enumeration of graphs representing the kinematic structure of mechanisms (18) (see Appendix 4).

Difficulties with an algebraic approach to the graph enumeration of mechanisms include the following:
(i) The problem of graph enumeration and the combinatorial analysis involved, is complex. Even in simple cases application of Polya's theory or its extension is often difficult.
(ii) Polya's Theorem predicts the number of distinct structures, but does not construct the individual structures.
(iii) In addition to the graph characteristics already mentioned governing the graphs of mechanisms, there are mechanical considerations, which result in further restrictions, often in the form of inequalities. We require, for example, that no part of a mechanism can be locked and that the mechanism should not be fractionated or have partial mobility (see T.H. Davies (10). For plane ten-bar linkages, for example, Woo (31) has shown that such requirements lead to an inequality of the form: $3l^* - 2j^* > 4$, where $l^*$, $j^*$ denote, respectively, the number of vertices and edges of an appropriately defined set of subgraphs of the graph of the mechanism.

In view of these difficulties, use has been made of algorithms (McLarnan (24), Woo (31)) and atlases of graphs listed by number of vertices and edges. These are available for graphs with up to 6 vertices [Buschbaum (4), DeBoer (12) and Harary (23)].
For mechanisms with kinematic pairs other than turning pairs, a number of enumeration studies have been made, both for plane and spatial mechanisms. In general, the greater the variety of kinematic pairs, the simpler the enumeration procedure. The most difficult case is the enumeration of the constrained bar linkages mentioned earlier. The case of linear mechanical systems (gear trains, gear transmissions and epicyclic drives) will be discussed separately in the next section.

(d) **Correspondence between graph structure and displacement equations.**

For purposes of motion analysis, it would be desirable to have a 1:1 correspondence between the displacement equations of a mechanism and its graph.

In principle, one can view this correspondence as follows. For each fundamental circuit, one can write a displacement equation, which expresses the loop closure of the corresponding independent mechanical circuit of the kinematic chain. Such equations can be conveniently expressed in vector form, matrix form or, in the case of plane motion, in complex-number form. For example for the four-bar linkage shown in Fig. 1b, there is one independent loop and the loop-closure expresses the fact that the vector sum of the four link lengths, regarded as vectors, vanishes.

In the case of plane linkage mechanisms, it is not difficult to devise algorithms, which yield the displacement equations directly from the graph structure (see, for instance, R.L. Dratch (14)). It is also not difficult to show that the number of equations obtained in this way matches exactly the number required to determine the displacement of the mechanism. Automatic, general-purpose computer programs in the mechanisms field (statics, kinematics, dynamics) essentially along these lines have been developed by Chace (3), Dratch (14), Paul (27), Uicker (30) and others.

The nature of the correspondence between the graph and the mechanism in the case of mechanisms other than pure linkages is not always as obvious.

For example, an apparent paradox was observed in the enumeration
of epicyclic gear trains (4). The graph shown in Fig. 3 apparently satisfies all the degree-of-freedom requirements of single-degree-of-freedom gear trains, having 5 links, 3 gear pairs and 4 turning pairs (P=2L-2j+1-3, where \( j \) denotes the number of gear pairs). Yet it is not the graph of a gear train. At the same time, it was found that for a given graph of a gear train, there might be more than one set of displacement equations. This seemed puzzling until it was found that due to mechanical considerations, the graph representation of mechanical systems, such as gear trains, must be further restricted as follows:

![Graph](image)

**Fig. 3:** Graph arising in the enumeration of gear trains.

(i) There can be no circuit formed exclusively by edges representing turning pairs (turning edges, for short). The circuit would then correspond to a set of pin-connected links. These would then either represent a locked structure, or otherwise the rotatability of the links would not be proportional, as well as being potentially limited in range.

(ii) All vertices must have at least one incident turning edge, since each gear is mounted on a bearing (turning pair/the axis of which coincides with the axis of the gear.

(iii) Each gear pair operates at constant center distance. This distance is maintained by an arm or carrier link, which is either directly paired to ground or connected to ground through a sequence of pin-connected links: hence the subgraph obtained after removal of the edges representing the gear pairs (geared edges, for short) cannot be disjoint.

It follows therefore that:

**Theorem 1:**

The subgraph of the graph of a gear train, formed by deletion of the geared edges, is a tree.

This is, therefore, a unique tree associated with each gear drive. The geared edges are the chords and each such chord defines a fundamental circuit of the system. This, then,
constitutes a unique set of fundamental circuits for the gear drive, each one representing a gear pair and the unique carrier, or arm, which maintains the center distance between the gears. The vertex representing the carrier link is called the transfer vertex. In other words, there is one and only one transfer vertex for each fundamental circuit.

If we label (color) each of the turning edges according to the axis of the turning pair it represents, then each fundamental circuit is characterized by the following:

Theorem 2: The turning edges in each fundamental circuit of the graph of a gear train constitute a two-colored path, the colors being separated by the transfer vertex.

Fig. 4 illustrates this fact with regard to the gear train shown in Fig. 1.

Fig. 4: Graph of Fig. 2 with coloring of edges added \((a, b)\). Principal geometrical element: axes \(a, b\) of turning pairs; vertex 2 is transfer vertex for fundamental circuits 3-1-2 and 3-4-1-2.

Thus, Theorem 2 imposes a logical condition on the graph, namely that it must be colorable in a certain way. The graph shown in Fig. 3 cannot be colored in accordance with Theorem 2 and hence, does not represent a gear train. The subject is discussed more fully with the aid of Boolean algebra in (19).

The second puzzling effect, the possible multiplicity of displacement equations for a given graph, is also explained by Theorems 1, 2, by noting that the graph-coloring constraint defined in Theorem 2 must be satisfied by the graph. When this is done, there is a 1:1 correspondence between the graph structure of gear trains and their displacement equations. This is discussed more fully in (19) in terms of an induced correspondence between the graph and the displacement equations (see Appendix 5).

The motion analysis of gear trains and epicyclic drives now follows particularly simply. With each fundamental circuit, we associate the number triplet \((i, j, k)\), where \(i, j\) represent the vertices incident at the geared edge and \(k\) represents the transfer vertex. The displacement equation associated with the fundamental
circuit \((ij)(k)\) is:

\[
f(i,j,k) = A_i W_i + A_j W_j + A_k W_k = 0 \quad (7a),
\]

where \(A_i + A_j + A_k = 0\),  \( (7b), \)

\(W_i, W_j, W_k\) denote the angular velocities of links \(i, j, k\):

\(A_i = 1,\) and \(A_j = -N_{ij}; \ N_{ij}\) represents the signed gear ratio between the gears on links \(i, j\) and is numerically equal to the ratio of the number of teeth on each gear.

The kinematics, dynamics, power flow and statics of plane and three-dimensional gear transmissions and gear drives have been developed from this point of view (14, 20, 21). R. L. Dratch (14) has also pointed out certain characteristics of the subgraph obtained from the graph of a gear train by deletion of the turning edges. Fig. 5 shows a listing of single-degree-of-freedom gear trains and their graphs.

The graph structure of linear mechanical systems, such as gear trains or differential levers, thus shows that the correspondence between graph structure and the displacement equations must include the special mechanical restrictions associated with the class of mechanism under consideration.

In recent times there have also been investigations involving the kinematic structure of gear differentials (Molian (25), Sanger (26)), which involve certain "type diagrams", which are essentially graphs. The kinematic synthesis of differential gear systems involves a different (but derivable) structure than that considered here. It may be called the external structure of a gear train, as it involves only the input/output shafts. It is possible to analyze such systems in terms of interconnections of suitably defined triangular subgraphs (unpublished studies of E. Soylemez and the first author).

(e) Automatic sketching and animation of mechanisms

One of the basic goals of mechanical design analysis can be stated as follows: given only the kinematic structure and relevant dimensions, is it possible to automate a (computerizable) procedure for the sketching and animation of mechanisms and mechanical systems?

One of the earliest general attempts, somewhat in this general direction, is associated with M.I.T.'s Project Sketchpad. In the case of mechanisms, it seems logical to start with the graph representation. One may be interested in techniques involving the animation of a particular mechanism, or one may consider general techniques applicable to a variety of different mechanisms. At the present time, individual mechanisms have generally been animated on an ad hoc basis. This may well be the most suitable and economical current procedure.
Fig. 5: Some single-degree-of-freedom gear trains and their graphs from: ASME Trans. 93B, Feb. 1971, ppl 176-182; courtesy of American Society of Mechanical Engineers.
Fig. 5 (Cont'd)
A more general scheme is that developed by R. Kaufman
(KYNSYN project - M.I.T.), which involves linkage synthesis via
Burmester theory.

Extending the theory developed by A.J. Goldstein at Bell
Telephone Laboratories, for the embedding of a graph in a plane,
L.S. Woo (32) has developed an efficient algorithm for the rep-
resentation of a planar graph by a set of non-crossing lines and
nodes. This is described in Appendix 6.

And finally, in an unpublished study (Project CASKADE-Columbia
University (1970)), a technique was developed for the automatic
sketching and enumeration of gear transmissions and drives as a
function only of kinematic structure. Figs. 6, 7, illustrate
successive "frames" of the motion of a fairly complex, epicyclic
drive, obtained in this way.

![Diagram](image)

Fig. 6 Kinematic structure of gear train (coupled
epicyclic drive) shown in Fig. 7.
Fig. 7: Frames of animation of coupled epicyclic drive.
These scattered investigations may suggest possibilities of an eventual, more general approach to this rather difficult problem.

4. **SOME CURRENT AND UNSOLVED PROBLEMS IN THE KINEMATIC STRUCTURE OF MECHANISMS**

This involves both entirely unsolved problems and improvements on partially solved problems. We list some of these briefly and in no particular order.

(i) Efficient methods for the enumeration of the non-equivalent structures of plane and spatial mechanisms. For example, an algebraic solution for the number and specification of the structure of plane, constrained 2n-bar linkages, where n is integral.

(ii) A graph representation of mechanisms, which is capable of analyzing mechanisms with fractionated and partial mobility. Considerable progress along these lines has already been made by Davies (10).

(iii) Improvements in the generality (versatility) and efficiency of computer-aided kinematic, static and dynamic analysis of mechanisms, given only the kinematic structure and essential dimensions: possible coordination with bond-graph theory.

(iv) A unified theory combining the Assur-type of mechanism classification with that obtained from the graph-theory approach to kinematic structure.

(v) Development of algorithms for the animation of mechanisms and mechanical systems, given only the kinematic structure.

(vi) A general theory for the existence, or nonexistence of a 1:1 correspondence between the graph of a mechanism and the displacement equations (see Appendix 5).

(viii) A graph representation of maximum utility for mechanisms with multiple joints.

(ix) A more efficient algorithm for the determination of isomorphism of the graphs of mechanisms.

(x) A more exhaustive treatment of the kinematic structure of mechanisms the graphs of which are non-planar. In what way, for example, would the characteristics of the independent mechanical circuits be affected? Mechanisms having non-planar graphs are not common, but do exist. For example, Fig.8 shows a ten-bar linkage having a non-planar graph. The linkage was shown by Gruebler (22).

(xi) Modification of the treatment of kinematic structure to include open kinematic chains and mechanisms derivable from these (e.g. gear differentials etc.). This involves a slight modification in the degree-of-freedom equations to allow for
links, which have only one element.

Fig. 8: Ten-bar linkage having non-planar graph.

(xii) Development of a system of mechanism classification, which could be used by the Patent Office. At the present time, patents are classified according to subject matter, e.g. hoisting machinery, baby carriages, typewriters etc. It is possible, however, and it is a recognized problem, that a mechanism used in one application may be patented in an entirely different area of application. This makes it difficult to determine whether a given mechanical device is novel or not. If an improved manner of characterizing mechanical structure could be found, perhaps it could eventually be incorporated in a cross-reference index by the Patent Office. Some of the techniques employed in artificial intelligence and heuristic searches may be useful in this connection.

(xii) Lastly we mention the interface between the above discussion of kinematic structure and cases in which this discussion is not applicable e.g. mechanisms the degree-of-freedom of which depends on metric considerations, non-rigid-body mechanical analysis, more general mechanical components and systems. This includes in part a definition of the region of applicability of the topological considerations of kinematic structure (which necessarily takes us outside the subject matter of this article) and in part a study of the foundations for the extension of the topological approach to more general systems.
References


Appendix 1

Freedom of unconstrained moving links = \lambda (l - 1)
Constraint exerted by i-th joint = (A_i - F_i)
Total constraint exerted by all joints = \sum_{i=1}^{\lambda} (A_i - F_i) = \lambda \sum F_i
Hence, F = \lambda (l - 1) = (A_j - \sum F_i) + \sum F_i

Appendix 2

Let \lambda (l-1) = number of available equations of statics.
U = No. of torque/force components, including external torques and forces.
P = total degree of freedom of all terminal joints.


P = No. of specified external torque/force reactions at 1st joint.
P = total no. of specified external torque/force components, assumed acting only at terminal joints.
\Sigma = summation symbol, referred to all joints; \sum refers to non-terminal joints and \sum refers to terminal joints.

An uncoupled non-terminal joint cannot support a torque/force component about/along an axis about/along which translation/rotation is permitted. At an uncoupled terminal joint, however, torque/force components can be sustained about each of the available axes. Summing the torque/force components over all joints, we have:

\Sigma = \lambda (l - 1) + \sum_{i=1}^{\lambda} (A_i - F_i) + \sum_{n=1}^{\lambda} (A_i - F_i) + \sum_{n=1}^{\lambda} = \lambda (l - 1) - P + F_T = \lambda (l - 1) - P + F_T

Equating E and U, we have:--

P = F_T - \{ \lambda (l - 1) + \sum F_i \}

= F_T - \sum \sum F_i

Appendix 3

Burnside's Lemma

Let S = \{a, b, ..., \} be a set of objects, and let G = \{g_1, g_2, ..., \} be a permutation group. An equivalence relation between elements of S is induced by G acting on S. The equivalence relation is defined by saying that a is equivalent to b if and only if there is a permutation, g, in G, which maps a into b. Combinatorial problems can often be reduced to a problem involving the determination of the number of such distinct equivalence classes. The enumeration of non-isomorphic graphs, which have a common contraction graph, is a typical example. W. Burnside has proven a lemma, which may be stated as follows:

The number of equivalence classes into which a set S is divided by the equivalence relation induced by a permutation group, G, is given by:

\frac{1}{|G|} \sum_{g \in G} \Psi(g),

where |G| is the number of permutations in the group G and \Psi(g) is the number of elements in S, which remain unchanged
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(or invariant) by the permutation g. Proof of the lemma is quite lengthy: it can be found in C.L. Liu: "Introduction to Combi-


Appendix 4

To state Polya's Theorem, some definitions are required:

Pattern: The functions equivalent under the group operation of

any element, g, of a permutation group G. G is typically

the group of automorphisms of a graph.

Patterns: the set of functions equivalent under the group operation

of an element, g, of a permutation group G.

Inventory: The product of the weights, \( w(r) \), where \( w(r) \) is a

weight associated with the element \( r \), belonging to the

set of elements \( r_1, \ldots, r_n \), of a set (store) \( R \).

Polya's Theorem:

The pattern inventory is obtained by substituting \( Z \left[ C \left( t_i \right)^j \right] \)

for \( t_i \) in the cycle index, \( C \), of the permutation group, \( G \),

where \( C \) is assumed expressed as a polynomial in the variables

\( t_i \).

A typical use of the theorem might be in enumerating the number

of (inequivalent) ways of coloring the edges or vertices of a cube

or a graph with a given number of colors.

In the case of constrained, plane, bar linkages, for example,

we wish to enumerate the number of non-isomorphic, linear graphs

with \( v \geq 4 \) vertices, \( \frac{1}{2} (3v - 4) \) edges, no vertex having a degree less

than 2 and no circuit of \( v' \) vertices having more than \( \frac{1}{2} (3v' - 4) \)

edges.

For mechanisms with single-loop graphs, the enumeration

problem is equivalent to the determination of the number of

necklaces with a given number of beads in each of several colors.

This special problem is solvable algebraically.

Appendix 5

We summarize some of the concepts for gear trains and gear

drives given in (19) by stating without proof one theorem and the

associated definitions.

Def. 1: Induced correspondence: a correspondence between the

links of two geared kinematic chains (kinematic chains

representing gear trains and gear drives) can also be

applied to the terms of their displacement equations by

associating the symbols for the vertices representing

a pair of meshing gears and the associated carrier, with

the motion variables associated with these links in the

corresponding displacement equation.

Def. 2: Isomorphic displacement equations: the displacement
equations of two gear trains are said to be isomorphic, if there is a 1:1 correspondence between their links, which induces a 1:1 correspondence between their displacement equations.

Def. 3: Rotation graph: The rotation graph of a gear train is defined as the graph obtained from the graph of the gear train by deleting the turning edges and the transfer vertices and labeling each geared edge according to the associated transfer vertex.

Def. 4: Isomorphism of rotation graphs: two rotation graphs are said to be isomorphic, if there is a 1:1 correspondence between their vertices and edges, which preserves incidence and labeling.

Theorem: If two gear trains have isomorphic rotation graphs, their rotational displacement equations are isomorphic.

Appendix 6
Algorithm for Straight-Line Representation of Simple Planar Graphs
Given a connection matrix of a simple planar graph, it is possible to draw a straight-line graph representation. An algorithm has been developed to solve this problem. Without loss of generality we may restrict our attention to simple planar graphs the nodes of which are of degree greater than two.

The construction procedure is outlined as follows:

Step 1: Determine the set of independent circuits, together with one dependent circuit, the collection of which defines regions in a plane. This step is based on a theory developed by A. J. Goldstein at Bell Laboratories (32). A more efficient algorithm has recently been developed by R. E. Tarjan of Stanford University [An efficient planarity algorithm", Stan-CS-244-71, Nov. 1971, Computer Science Department, Stanford University].

Step 2: Selecting an arbitrary circuit from the collection, preferably one with a large number of nodes, one can order the set of circuits in such a fashion that they may be drawn in sequence inside the selected circuit as the external boundary.

Step 3: Starting from the innermost circuit of step 2, construct a straight-line graph by successively assigning coordinates to the nodes. The nodes are processed in a reverse order from the sequence obtained in step 2. The mathematical principle of the separation theorem in linear vector space is utilized in the heuristic algorithm for assigning coordinates to the nodes.

The detailed procedure may be found in [32].