*Mechanism and Machine Theory*, 1972, Vol. 7, pp. 263–275. Pergamon Press. Printed in Great Britain

# Kinematics and Statics of a Coupled Epicyclic Spur-Gear Train

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and
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Received 22 September 1971

**Abstract**
The purpose of this paper is essentially tutorial: to help recast the classical textbook chapter on gear trains in a modern vein. Hopefully this involves general methods, avoiding ad-hoc reasoning and unnecessary complexity. We believe that recently developed methods [1, 2, 4], employing a graph representation of kinematic structure, are suitable for this purpose. The procedures for kinematic analysis, force analysis, and power-flow determination are outlined in simple step fashion, using, for purposes of illustration, a fairly complex, coupled epicyclic drive shown by Glover [5]. It is hoped that the simplicity of the method will commend itself both to the engineering student as well as to the practicing engineer.

**Zusammenfassung**—Kinematik und Statik eines gekoppelten epizyklischen Zahnradgetriebes: F. Freudenstein und A. T. Yang.
Der Zweck dieses Aufsatzes ist ein vorwiegend methodischer: er soll helfen, die klassischen Lehrbuchabschnitte über Rädertriebe in eine moderne Fassung umzuformen. Dies bedingt allgemeine Methoden unter Vermeidung unmotivierter Vorgangsweisen und unnötiger Verwicklungen. Wir glauben, daß die kürzlich entwickelten Methoden [1, 2, 4], welche kinematische Strukturen durch Graphen darstellen, für diesen Zweck geeignet sind. Die Verfahrensweisen zur kinematischen Analyse, zur Bestimmung der Kräfte und zur Ermittlung des Kraftflusses werden stufenweise auseinandergesetzt, wobei zur Erläuterung ein erträglich komplexes, von J. H. Glover [5] betrachtetes gekoppeltes Planetenradgetriebe herangezogen wird. Es ist zu hoffen, daß die Einfachheit der Methode sowohl den Ingenieurstudenten als auch den praktizierenden Ingenieur ansprechen wird.

**Резюме** — Кпнематика и статика эпициклической передачи с цилиндрическими прямозубыми колесами. Ф. Фройденштейн и А. Янг.
Цель этой статьи — это по существу консультация: помочь написать заново в современном духе главу классического учебника, посвященную зубчатым передачам. Это влечет за собой применение общих методов и избегание специального рассуждения и ненужной сложности. Мы уверены, что недавно развитые методы [1, 2, 4], применяющие диаграммное изображение кинематической структуры, соответствуют этой цели. Намечены методики кинематического анализа, анализа сил и определения потока мощности с применением для иллюстрации достаточно сложной эпициклической передачи, описанной Дж. Гловером [5].

Авторы надеются, что простота метода будет полезна студентам технических заведений и профессиональным инженерам.

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## 1. Introduction

In recent years, there has been an upsurge of interest in planetary gear trains. The many valuable publications on the subject [1–11] (we can list only very few†) recognize the need for a systematic development. The following is dedicated towards the same goal. The emphasis is on generality with a minimum of human decision-making, thereby freeing the engineer for the more creative aspects of gear design and resulting in methods which are suitable for automatic computation. It is hoped also that the engineering student, who has too often in the past been repelled by the archaic nature of the usual textbook treatment of the subject, may find some challenge in the present approach.

## 2. Graph Representation of Kinematic Structure

Essentially this is a schematic diagram which answers the question: "which link is connected to which other link by what kind of a joint (pair)". More precisely, we have:

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*Step* 1: *Kinematic Structure*

*Number each link (1, 2, 3, . . .).
*Label axes of turning pairs (a, b, c, . . .).
*Represent each link by a correspondingly numbered point (vertex).
*A gear mesh between two links is represented by a heavy line (geared edge) connecting the corresponding vertices.
*A turning pair between two links is represented by a light line (turning edge) connecting the corresponding vertices: Label each turning edge according to its pair axis (a, b, c, . . .).
*Identify the fixed link by drawing a small circle around the corresponding vertex

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The diagram defined in Step 1 is called a *graph*. The size and shape of the diagram are immaterial; only the topology matters.

Figure 1a shows a functional schematic of a coupled, epicyclic drive shown in Glover[5] as Drive # 11. In the following we shall refer to this drive simply as Glover # 11. Each link has been numbered and the different axes of the turning pairs identified.

Figure 1b shows the graph of the mechanism. As will be discussed later, when several planet gears are in parallel (e.g. 5 and 5′ or 6 and 6′) only one is shown in the graph. The others are kinematically redundant, but serve to distribute the load.

The graph has certain characteristics, which are useful in analyzing gear trains and which may also be regarded as checks on the accuracy of Step 1:

*Check* #1. The diagram (subgraph), consisting only of the turning edges and their endpoints, contains all the vertices of the graph, but no circuits (such a graph is called a *tree*).

This is due to the constancy of the center distance between gears in mesh and the potentially unlimited, proportional rotations of all links. This implies that each gear has a turning pair concentric with its axis and that there can be no circuit involving turning pairs only.

In Fig. 1b, this subgraph (tree) consists of all turning edges, e.g. edges (35), (13), (14), (46), (42) and the 6 vertices.

†An extensive bibliography is given by Z. Levai: *Bibliography of planetary mechanism*, BME, Budapest, 1969.

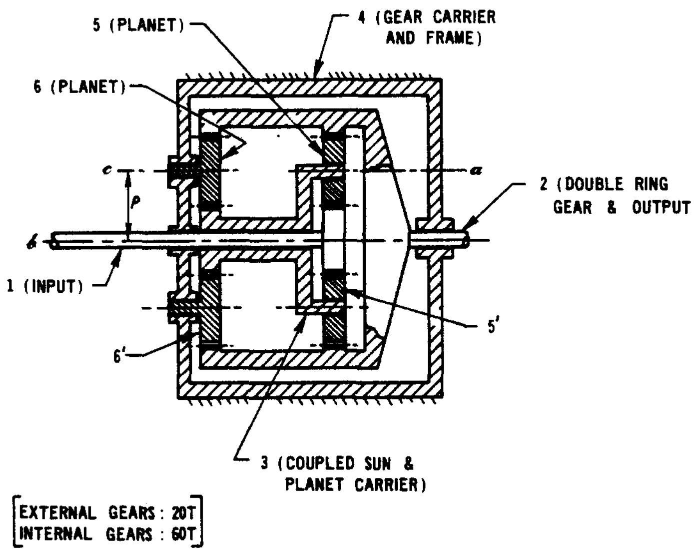


**Figure 1a.** Functional schematic of coupled epicyclic drive indicated as Drive #11 in Ref. [5] (Glover).

*Check #2*. Each geared edge can be associated with a fundamental circuit of the graph.

We shall call these fundamental circuits *f-circuits* for short. Each *f*-circuit consists of one geared edge and the turning edges connecting the endpoints of the geared edge. There is only one way of selecting these turning edges, if the convention is adopted that when traversing the edges of the tree to go from one end-point to the other, no turning edge may be retraced (in the language of graph theory, the tree path is unique).

For example, in Fig. 1b, the *f*-circuits are:

Circuit I: (15) (53) (31)
Circuit II: (25) (53) (31) (14) (42)
Circuit III: (62) (24) (46)
Circuit IV: (36) (64) (41) (13)

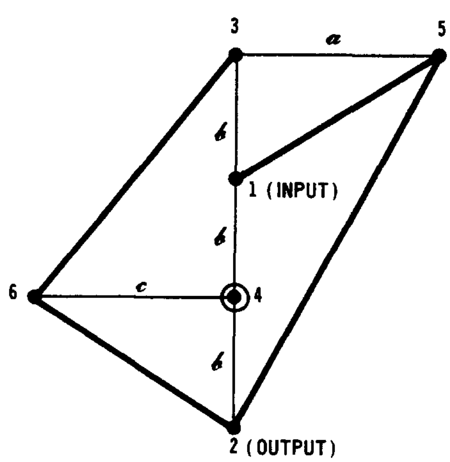


**Figure 1b.** Graph of epicyclic drive shown in Fig. 1a.

*Check #3*. In each fundamental circuit, there is exactly one vertex, the turning edges incident at which represent different pair axes. It is called the *transfer vertex*.

In each gear mesh, the gear carrier (also called the arm) maintains the constancy of the center distance between the gears. The vertex representing the gear carrier is incident at turning edges representing the pair axes of the meshing gears. This vertex is the transfer vertex.

In Fig. 1b, the transfer vertices are as follows:

Circuit I: Vertex 3 (pair axes a, b)
Circuit II: Vertex 3 (pair axes a, b)
Circuit III: Vertex 4 (pair axes b, c)
Circuit IV: Vertex 4 (pair axes b, c)

*Check #4*. The diagram (subgraph) consisting only of the geared edges and their endpoints may have no circuits.

Otherwise, special dimensions would be required for movability of the mechanism. Although such cases can occur, they are excluded in the present analysis. Circuits with only geared edges also can point to the existence of planet gears in parallel. For example, if gears 5′ and 6′ had been included in the graph shown in Fig. 1b, the circuits (3-6-2-6′-3) and (1-5-2-5′-1) would have involved geared edges only. We therefore omit all but one of the identical gears (planets), the axes of which are turning-pair connected to the same member.

*Check #5*. All turning edges having identical pair axes must be connected (and can have no circuits).

Otherwise the center distances could not remain constant. In Fig. 1b, these are edges (31) (14) (42).

*Check #6*. The degree-of-freedom equations for the gear train are:

$$l = j_T + 1$$
$$j_T = j_G + F$$
$$L = j_G$$

where $l$ = number of links, $j_T$ = number of turning pairs, $j_G$ = number of gear pairs, $L$ = number of fundamental circuits and $F$ = degree of freedom of gear train.

In Fig. 1b, for example, $l = 6, j_T = 5, j_G = L = 4$ and $F = 1$. We conclude that this is a single-degree-of-freedom gear train.

The first equation is obtained directly from the basic tree property: number of vertices of tree exceeds number of edges by unity. The other equations follow from the first equation and the freedom equations for mechanisms.

We note that in the graph representation of kinematic structure, we do not distinguish between internal and external gears and several gears may be part of one rigid link, and thus are represented by a single vertex.

## 3 Kinematic Analysis

We outline the procedure in terms of several steps.

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*Step 2: Angular-Velocity Equations*

*Identify each fundamental circuit symbolically as $(ij)$ $(k)$, where $i, j$ represent the links containing the gears in mesh and $k$ the gear carrier (transfer vertex).

*For each fundamental circuit. The angular-velocity equation is:

$$\omega_i - \omega_j N_{ji} + \omega_k (N_{ji} - 1) = 0 \tag{3.1}$$

where $\omega_i$, $\omega_j$, $\omega_k$, denote the angular velocities of links $i$, $j$, $k$ and $N_{ji} = \pm T_j / T_i$, where $T_i$, $T_j$ denote the number of teeth on gears $i$, $j$, respectively, and the ratio is positive or negative according as the gear mesh is internal or external

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There are as many equations of type (3.1) as there are gear meshes (or $f$-circuits). The equation is readily derived by considering the motion relative to the arm:

$$(\omega_i - \omega_k)/(\omega_j - \omega_k) = N_{ji}.$$

In Glover's #11 Drive (Fig. 1b), the angular-velocity equations are:

| *Fundamental circuit* | *Angular-velocity equations* | |
|---|---|---|
| Circuit I: (15) (3) | $\omega_1 - \omega_5 N_{51} + \omega_3 (N_{51} - 1) = 0$ | (3.2) |
| Circuit II: (25) (3) | $\omega_2 - \omega_5 N_{52} + \omega_3 (N_{52} - 1) = 0$ | |
| Circuit III: (26) (4) | $\omega_2 - \omega_6 N_{62} + \omega_4 (N_{62} - 1) = 0$ | |
| Circuit IV: (36) (4) | $\omega_3 - \omega_6 N_{63} + \omega_4 (N_{63} - 1) = 0$ | |

*Check* #1. The sum of the coefficients of the angular velocities in each equation is zero.

*Check* #2. Balance number of equations vs. number of unknowns.

In Glover's #11 Drive, link 4 is fixed, so that $\omega_4 = 0$. Link 1 is the input link, the angular velocity of which, $\omega_1$, is assumed known. This leaves 4 equations (equations (3.2)) in the 4 unknowns: $\omega_2$, $\omega_3$, $\omega_5$, $\omega_6$.

The angular-velocity equations are linear, simultaneous algebraic equations. Their solution is outlined in the next step.

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*Step* 3: *Solution of Angular-Velocity Equations*

*Solve the angular-velocity equations for the unknown angular velocities

*Solution for single-degree-of-freedom gear trains based on Cramer's rule:

Let $A_{mn}$ be the value of the determinant of the square matrix obtained by deleting the $m^{th}$ and $n^{th}$ columns of the coefficient matrix of the angular-velocity equations, including the terms containing the fixed link, $f$.

Then the angular-velocity ratio of links $p$, $q$ is given by:

$$\frac{\omega_p}{\omega_q} = (-1)^{p+q} \frac{(p-f)(q-f)}{|p-f| \cdot |q-f|} \left[ \frac{A_{pf}}{A_{qf}} \right] \tag{3.3}$$

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For example, in the Glover #11 Drive, the $(4 \times 6)$ coefficient matrix is:

$$\begin{pmatrix} 1 & 0 & N_{51}-1 & 0 & -N_{51} & 0 \\ 0 & 1 & N_{52}-1 & 0 & -N_{52} & 0 \\ 0 & 1 & 0 & N_{62}-1 & 0 & -N_{62} \\ 0 & 0 & 1 & N_{63}-1 & 0 & -N_{63} \end{pmatrix}$$

For $p = 1, q = 2, f = 4$, we have

$$A_{14} = \begin{Vmatrix} 0 & N_{51}-1 & -N_{51} & 0 \\ 1 & N_{52}-1 & -N_{52} & 0 \\ 1 & 0 & 0 & -N_{62} \\ 0 & 1 & 0 & -N_{63} \end{Vmatrix} = -N_{51}N_{62} - N_{63}(N_{52} - N_{51}).$$

$$A_{24} = \begin{Vmatrix} 1 & N_{51}-1 & -N_{51} & 0 \\ 0 & N_{52}-1 & -N_{52} & 0 \\ 0 & 0 & 0 & -N_{62} \\ 0 & 1 & 0 & -N_{63} \end{Vmatrix} = N_{52}N_{62}.$$

From Equation (3.3) we obtain the angular velocity ratio

$$\frac{\omega_1}{\omega_2} = (-1)^3 \frac{(-3)\ (-2)}{3 \times 2} \left[\frac{A_{14}}{A_{24}}\right] = 1 - \left(\frac{N_{51}}{N_{52}} - 1\right)\left(\frac{N_{63}}{N_{62}} - 1\right) = R \text{ (say)}.$$

Similarly, we find that:

$$\left.\begin{aligned} &\omega_2/\omega_1 = 1/R \\ &\omega_3/\omega_1 = N_{63}/(N_{62}R) \\ &\omega_5/\omega_1 = [1 - (1 - N_{52})(N_{63}/N_{62})]/(RN_{52}) \\ &\omega_6/\omega_1 = 1/(RN_{62}). \end{aligned}\right\} \tag{3.4}$$

The gear ratios given in Glover's #11 Drive are as follows:

$$N_{51} = -20/20 = -1; \quad N_{62} = 20/60 = \tfrac{1}{3}$$
$$N_{52} = 20/60 = \tfrac{1}{3}; \quad N_{63} = -20/20 = -1.$$

This gives the following angular-velocity ratios:

$$\omega_1 : \omega_2 : \omega_3 : \omega_4 : \omega_5 : \omega_6 : = 15 : -1 : 3 : 0 : -9 : -3$$

Since the angular-velocity equations are unaffected if each angular velocity is increased by the same constant, we can find the angular velocities when a link other than link 4 is held fixed, by adding to each angular velocity a constant equal to the negative of the angular-velocity of the new fixed link, as shown in the following table:

| Fixed link | Angular-velocity ratios | | | | | |
|---|---|---|---|---|---|---|
| | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\omega_4$ | $\omega_5$ | $\omega_6$ |
| 1 | 0 | −16 | −12 | −15 | −24 | −18 |
| 2 | 16 | 0 | 4 | 1 | −8 | −2 |
| 3 | 12 | −4 | 0 | −3 | −12 | −6 |
| 4 | 15 | −1 | 3 | 0 | −9 | −3 |
| 5 | 24 | 8 | 12 | 9 | 0 | 6 |
| 6 | 18 | 2 | 6 | 3 | −6 | 0 |

## 4. Static Force and Torque Analysis

We consider the case without friction. Before deriving the equations of equilibrium, we consider the forces and torques associated with a fundamental circuit: $(ij)$ $(k)$. Details of the derivations can be found in Appendix I.

*Center distance.* $\mathbf{a}_{ij} = a_{ij}\hat{\mathbf{a}}_{ij}$ denotes the vector distance from the axis of gear $i$ to the axis of gear $j$; $a_{ij} = a_{ji}$ and $\hat{\mathbf{a}}_{ij}$ is a unit vector in the direction of $\mathbf{a}_{ij}$.

*Unit normal.* $\mathbf{n}$ represents a unit vector in the direction of a positive angular-velocity vector.

*Reactions associated with geared edge, ij*

$$\mathbf{F}_{ij} = \text{tangential force exerted by gear } i \text{ on gear } j = F_{ij}(\hat{\mathbf{a}}_{ij} \times \mathbf{n}), \text{ with } F_{ij} = F_{ji} \tag{4.1a}$$

$$\mathbf{T}_{ij} = \text{torque exerted by } \mathbf{F}_{ij} \text{ about axis of gear } j = T_{ij}\mathbf{n}, \text{ where } T_{ij} = (a_{ij}F_{ij})/(1 - N_{ij}). \tag{4.1b}$$

*Reactions associated with turning edge, ik*

$$\mathbf{F}_{ik} = \text{tangential force exerted by gear } i \text{ on carrier } k = F_{ik}(\hat{\mathbf{a}}_{ij} \times \mathbf{n}), \text{ with } F_{ik} = -F_{ki} \tag{4.1c}$$

$$\mathbf{F}_{ki} = \text{force exerted by carrier } k \text{ on gear } i = -\mathbf{F}_{ik} \tag{4.1d}$$

$$\mathbf{T}_{ik} = \text{torque exerted by } \mathbf{F}_{ik} \text{ about fixed axis of gear carrier } k = T_{ik}\mathbf{n}, \text{ where } T_{ik} = \rho F_{ik} \tag{4.1e}$$

and $\rho$ is the distance between the fixed pair axis of link $k$ and the pair axis represented by edge $ik$ (i.e. $\rho$ is either equal to $a_{ij}$ or 0).

$$\mathbf{T}_{ki} = \text{torque exerted by force } \mathbf{F}_{ki} \text{ about axis of gear } i = 0. \tag{4.1f}$$

*External torques*

$$\mathbf{T}_p = T_p\mathbf{n} \text{ represents the external torque acting on link } p. \tag{4.1g}$$

Then in the case in which there are no floating arms and in each gear mesh one pair axis is fixed, the force analysis can be carried out as follows:

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*Step 4: Force and Torque Determination*

*Associate a force, $\mathbf{F}_{pq}$, with edge $pq$. This is the force exerted by link $p$ on link $q$ through the kinematic pair, $pq$.

*Associate a torque, $\mathbf{T}_{pq}$, with each force, $\mathbf{F}_{pq}$, as in equations (4.1).

*For each moving link $(q)$ having a fixed axis, write a scalar torque-balance equation:

$$\sum_p T_{pq} + T_q = 0. \tag{4.2a}$$

For $T_{pq}$ use the expressions given in equations (4.1b, e, f).

*For each floating link $(q)$ (link without a fixed axis), write one scalar torque-balance equation and one scalar force-balance equation:

$$\sum_p T_{pq} + T_q = 0 \tag{4.2b}$$

$$\sum_p F_{pq} = 0. \tag{4.2c}$$

For $T_{pq}$ use the expressions given in equations (4.1b, e, f).

*Solve equations (4.2a, b, c) for $F_{pq}$, $T_{pq}$ and $T_q$ as a system of $2(v-1)-e_f$ equations in as many unknowns, where $v$ denotes the number of vertices of the graph, and $e_f$ the number of turning edges with fixed pair axes. It is assumed that all but one of the external torques are given.

*Find the vectors $\mathbf{F}_{pq}$, $\mathbf{T}_{pq}$ from equations (4.1).

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Proof of the determinacy of this procedure is given in Appendix II.

We continue with Glover's #11 Drive as illustration. We assume a known input torque, $T_1\mathbf{n}$, on link 1; $T_2\mathbf{n}$ is the load torque on link 2.

The unknowns are: $F_{15}$, $F_{35}$, $F_{25}$, $F_{36}$, $F_{62}$, $T_2$.

Links with fixed axis: 1, 2, 3, 6 (one torque equation each).

Floating link: 5 (one torque and one force equation).

The equations of statics are:

*Link* 1: $T_1 + T_{51} = 0$;

or

$$T_1 + \frac{a_{51}F_{51}}{(1-N_{51})} = 0 \tag{4.3a}$$

*Link* 2: $T_2 + T_{52} + T_{62} = 0$;

or

$$T_2 + \frac{a_{52}F_{52}}{(1-N_{52})} + \frac{a_{62}F_{62}}{(1-N_{62})} = 0 \tag{4.3b}$$

*Link* 3: $T_{53} + T_{63} = 0$;

or

$$a_{15}F_{53} + \frac{a_{63}F_{63}}{(1-N_{63})} = 0 \tag{4.3c}$$

*Link* 4: Fixed link; no equation necessary.

*Link* 5: $T_{15} + T_{35} + T_{25} = 0$;

or

$$\frac{a_{15}F_{15}}{(1-N_{15})} + 0 + \frac{a_{25}F_{25}}{(1-N_{25})} = 0 \tag{4.3d}$$

$$F_{15}+F_{35}+F_{25}=0 \tag{4.3e}$$

*Link* 6: $T_{36}+T_{26}=0$;

or

$$\frac{a_{36}F_{36}}{(1-N_{36})}+\frac{a_{26}F_{26}}{(1-N_{26})}=0. \tag{4.3f}$$

For the proportions given by J. H. Glover, we have

$$a_{51}=a_{52}=a_{62}=a_{63}=\rho \text{ (say)}.$$

Using these proportions and equations (4.1), the solution to equations (4.3) may be expressed as

$$\mathbf{F}_{51}=(-T_1/\rho)(1-N_{51})(\hat{\mathbf{a}}_{51}\times\mathbf{n}) \tag{4.4a}$$

$$\mathbf{F}_{52}=(T_1/\rho)(N_{51}/N_{52})(\hat{\mathbf{a}}_{52}\times\mathbf{n}) \tag{4.4b}$$

$$\mathbf{F}_{53}=(T_1/\rho)[1-(N_{51}/N_{52})](\hat{\mathbf{a}}_{51}\times\mathbf{n}) \tag{4.4c}$$

$$\mathbf{F}_{63}=(-T_1/\rho)(1-N_{63})[1-(N_{51}/N_{52})](\hat{\mathbf{a}}_{63}\times\mathbf{n}) \tag{4.4d}$$

$$\mathbf{F}_{26}=(T_1/\rho)(N_{63}/N_{62})(1-N_{62})[1-(N_{51}/N_{52})](\hat{\mathbf{a}}_{26}\times\mathbf{n}). \tag{4.4e}$$

The torque, $T_2$, is given by equation (4.3b):

$$T_2=-\rho\left[\frac{F_{52}}{(1-N_{52})}+\frac{F_{62}}{(1-N_{62})}\right]$$

which, with the aid of equations (4.4b, e), gives

$$-\left(\frac{T_2}{T_1}\right)=1-\left(\frac{N_{51}}{N_{52}}-1\right)\left(\frac{N_{63}}{N_{62}}-1\right)=R \tag{4.4f}$$

*Check* #1: All fixed reactions are omitted; these can be obtained separately, if desired.

*Check* #2: All coefficients of the equilibrium equations are constants.

*Check* #3: Balance number of equations vs. number of unknowns.

For example, in Glover's #11 Drive, $v=6$ and $e_f=4$ (edges 31, 14, 64, 42). Hence, the number of equations and of unknowns should equal $2(6-1)-4=6$.

*Check* #4: Check that power INTO mechanism = power OUT of mechanism.

In Glover's #11 Drive, combining equation (4.4f) with the first of equations (3.4), we have:

$T_1\omega_1+T_2\omega_2=0$, which checks the power flow into and out of the mechanism.

## 5. Power Flow

We consider the power flow in the absence of energy losses due to friction. It is convenient to associate a power flow, $P_{pq}$, with edge $pq$ as follows: $P_{pq}$ = power transmitted by link $p$ to link $q$ across kinematic pair, $pq$ in the direction of $p$ to $q$.

Then

$$P_{pq} = \mathbf{F}_{pq} . \mathbf{V}_{pq} \tag{5.1}$$

where $\mathbf{F}_{pq}$ is the transmitted force across the pair represented by edge $pq$, as defined previously [equation (4.1)], and $\mathbf{V}_{pq}$ denotes

(i) the linear velocity of the pair axis of pair $pq$, if $pq$ is a turning pair;
(ii) the velocity of the pitch point of gears $p$ and $q$, if $pq$ is a gear pair.

Limiting ourselves once again to the case in which there are no floating arms and in which there is a fixed pair axis in each gear mesh, it is readily shown, using the methods of Appendix I, that:

$$\text{Case (i):} \quad P_{pq} = -F_{pq}a_{qr}\omega_p = -P_{qp} \tag{5.2a}$$

where edge $pq$ lies in fundamental circuit $(qr)(p)$, so that $p$ represents the gear carrier (axis of gear $r$ is fixed).

$$\text{Case (ii):} \quad P_{pq} = -\frac{F_{pq}a_{pq}\omega_p}{1-N_{qp}} = -P_{qp} \tag{5.2b}$$

where $p$ denotes the link with the gear having a fixed pair axis.

If $P_{pq}$ is positive, the power flow is directed from link $p$ to link $q$; if $P_{pq}$ is negative, the power flow is directed from link $q$ to link $p$. The calculations are outlined in the following step:

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*Step 5: Power Flow*

*Associate a directed power flow, $P_{pq}$, with each edge, the assigned direction for positive $P_{pq}$ being from link $p$ to link $q$.
*At any vertex the flow of power into the vertex is equal to the flow of power away from the vertex ("current law").
*At any vertex incident at more than 2 edges representing pairs in which both elements are moving, the power *branches*.
*Calculate the power flow in each branch. Use equation (5.2a) for power flow across a turning edge. Use equation (5.2b) for power flow across a geared edge ("resistance law"). Generally:

$$P_{pq} = \mathbf{F}_{pq} . \mathbf{V}_{pq}$$

*Continue until entire power flow is determined.

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For example, in Glover's #11 Drive, the power input ($P_1$) is $T_1\omega_1$. Applying the current law to vertex 1, we have:

$$P_{15} = P_1.$$

At vertex #5 there is a branching of power:

$$P_{52} = \mathbf{F}_{52} . \mathbf{V}_{52}. \tag{5.2b}$$

Since edge 52 is a geared edge, use equation (5.2b) for the power flow:

$$P_{52} = \frac{F_{52}a_{52}\omega_2}{(1-N_{52})} \qquad (p = 2).$$

Substituting for $F_{52}$, from equation (4.4b), we obtain:

$$P_{52}/P_1 = N_{51}/(RN_{52}) = 0{\cdot}2.$$

Hence, the power flow across edge 52 is directed from link 5 to link 2 and is equal to 20 per cent of input power.

*Check* #1: Power flow across a kinematic pair, one of whose elements is fixed, is zero.

In Glover #11, the pair axes of pairs 13, 14, 64, 42 are in this category. Hence, $P_{13} = P_{14} = P_{64} = P_{42} = 0$.

*Check* #2: Power flow across vertices of degree 2 (incident at two edges): power IN equals power OUT (fixed edges do not count in this check).

In Glover #11, this implies that $P_{53} = P_{36} = P_{62} = (0{\cdot}8)P_1$. This also shows that the power balance for vertex 2 gives $P_1 = P_2$, where $P_2$ represents the power flow out of the mechanism.

*Check* #3: the power flows are constant multiples of the input power. If the angular-velocities of the links are constant, the power flows are, therefore, constant.

## 6. Conclusion

Simple, general programmable procedures have been described for the kinematic analysis, static force analysis, and power flow without friction, of spur gear epicyclic gear trains, directly from the kinematic structure. In order to illustrate the general approach, we have limited ourselves to the simpler aspects of gear-train design. The inclusion of friction, elastic effects, dynamic considerations, floating arms, and singular configurations can be developed along similar lines, but would take us beyond the scope of this introductory exposition.

*Acknowledgements*—The authors are grateful to the Army Research Office (Durham) for the support of this research through Contract #DA-31-124-ARO(D)-178. We should also like to express our gratitude to the participants in the NSF Advanced Training Workshop in Mechanisms at Oklahoma State University, Stillwater, Oklahoma, 26 July–6 August 1971. Their questions and comments in the course of a presentation of the main ideas of this approach led to a number of useful changes and improvements.

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## Appendix 1: Static Torque and Force Analysis

Referring to Fig. 2, the geometry is as follows:

$$\mathbf{r}_i - \mathbf{r}_j = a_{ij}\hat{\mathbf{a}}_{ij} \quad (A1)$$

$$r_i/r_j = N_{ij} = 1/N_{ji} \quad (A2)$$

$$\mathbf{r}_i = N_{ij}\mathbf{r}_j \quad (A3)$$

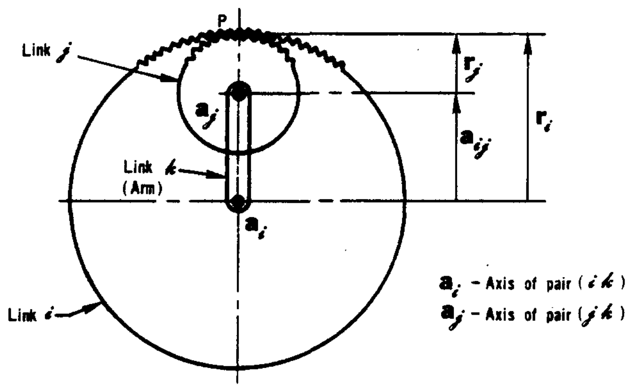


**Figure 2.** Geometry of a simple gear mesh.

Hence,

$$\mathbf{r}_i = [N_{ij}a_{ij}/(N_{ij}-1)]\hat{\mathbf{a}}_{ij} \quad (A4)$$

$$\mathbf{r}_j = [a_{ij}/(N_{ij}-1)]\hat{\mathbf{a}}_{ij} \quad (A5)$$

where

$$\mathbf{a}_{ij} = a_{ij}\hat{\mathbf{a}}_{ij} \quad (A6)$$

$$a_{ij} = a_{ji} \quad (A7)$$

$$\hat{\mathbf{a}}_{ij} = -\hat{\mathbf{a}}_{ji}. \quad (A8)$$

With the unit normal, $n$, in the direction of a positive angular-velocity vector, the forces and torques are as follows:

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji} = F_{ij}(\hat{\mathbf{a}}_{ij}\times\mathbf{n}) \quad (A9)$$

$$F_{ij} = F_{ji} \quad (A10)\dagger$$

$$\mathbf{T}_{ij} = \mathbf{r}_j \times \mathbf{F}_{ij} = T_{ij}\mathbf{n} \quad (A11)$$

$$T_{ij} = a_{ij}F_{ij}/(1-N_{ij}) \quad (A12)$$

$$\mathbf{F}_{ik} = F_{ik}(\hat{\mathbf{a}}_{ij}\times\mathbf{n}) \quad (A13)$$

$$\mathbf{F}_{ki} = F_{ki}(\hat{\mathbf{a}}_{ij}\times\mathbf{n}) \quad (A14)$$

$$F_{ik} = -F_{ki} \quad (A15)\dagger$$

$$\mathbf{T}_{ik} = \mathbf{a}_{ji}\times\mathbf{F}_{ik} = T_{ik}\mathbf{n} \quad (A16)$$

$$T_{ik} = \rho F_{ik} \quad (A17)$$

$$\left.\begin{array}{l}\rho = a_{ij}\ (\text{if } a_j \text{ is fixed})\\ \quad\text{or } 0\ (\text{if } a_i \text{ is fixed})\end{array}\right\} \quad (A18)$$

$$\mathbf{T}_{ki} = 0 \quad (A19)$$

## Appendix II: Proof of Determinacy of Force-Analysis Procedure

No. of equations = No. of moving links ($v-1$) plus no. of floating links ($l_f$).

But $l_f$ = No. of moving links less number of vertices ($v_f$) incident at edges with fixed-location (turning) pair axes, excluding the vertex representing the fixed link.

†In order to satisfy Newton's 3rd Law (Action and reaction are equal and opposite).

Let $e_f$ denote the number of edges with fixed (turning) pair axes. The subgraph consisting of the edges $e_f$ and their endpoints must be connected, but cannot have any circuits. Hence, as in the case of trees, the number of vertices in the subgraph equals $(e_f + 1)$. But the number of vertices is equal to $(v_f + 1)$.

Hence, $v_f = e_f$ and $l_f = (v-1) - e_f$. Hence, the number of equations is equal to $2(v-1) - e_f$. The number of unknowns = Total number of edges less number of edges with fixed pair axis plus 1 (unknown torque) = $e - e_f + 1$.

For single-degree-of-freedom trains, $e = 2(v-1) - 1$. Hence, the number of equations is equal to the number of unknowns. Provided that the equations are independent, the procedure is, therefore, determinate. We assume also that all but one of the external torques (usually the output torque) are known.

When the degree of freedom of the gear train is more than one, the argument is similar, provided we assume that the number of unknown external torques to be equal to the degree of freedom of the train.

In the case of gear differentials, these conclusions need to be modified, because of the presence of a link (the fixed link) with but a single pair axis which modifies the degree-of-freedom equation. It is not difficult to do so, but this would take us beyond the scope of this simple exposition.