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Synthesis of Quasi-Constant Transmission Ratio Planar Linkages

The present paper deals with the formulation of novel closed-form algorithms for the kinematic synthesis of quasi-constant transmission ratio planar four-bar and slider–crank linkages. The algorithms are specific for both infinitesimal and finite displacements. In the first case, the approach is based on the use of kinematic loci, such as centrodes, inflection circle, and cubic of stationary curvature, as well as Euler–Savary equation. In the second case, the design equations follow from the application of Chebyshev min–max optimality criterion. These algorithms are aimed to obtain, within a given range of motion, a quasi-constant transmission ratio between the driving and driven links. The numerical examples discussed allow a direct comparison of structural errors for mechanisms designed with different methodologies, such as infinitesimal Burmester theory and the Chebyshev optimality criterion. [DOI: 10.1115/1.4031058]

1 Introduction

Planar linkages have wide applications in the engineering practice for the generation of prescribed nonuniform motions in automatic machinery. According to motion requirements, three different categories are traditionally identified [1–7]: (a) function generators, when a relative motion kinematic law between input–output links is assigned [8]; (b) rigid body guidance, when a link of the mechanism is constrained to occupy a series of given positions; and (c) path generators, when an curve or a set of points of such curve are to be approximated by means of a coupler curve. The typical procedure to develop a systematic mechanism design consists of three main steps: task definition, type synthesis, and dimensional synthesis [9–17].

Periodic motions with uniform transmission ratio in a prescribed range are required in several engineering applications, such as automatic machines. Gears and cams can satisfy this design specification accurately, but they are more expensive and difficult to manufacture than linkages, and not always suitable, or convenient, to transmit the motion between two parallel axes with large center distance. Thus, when a quasi-constant transmission ratio within a wide range of driving link rotation is allowed, four-bar linkages are candidates to replace circular gears or cams. One design method was proposed and reported in his textbook by Hall [18]. He used the inflection circle and the polodes curvature for the relative motion between the driving and driven links. The polodes and their evolutes of slider–crank/rocker mechanisms were analyzed in depth in Ref. [19]. This approach was also cited by Hartenbertg and Denavit in Ref. [20], who solved the same problem by using the cubic of stationary curvature and, likewise, by Hain and Marx [21], who made use of the Freudenstein theorem. An interesting application of constantly transmitting four bars to a translating platform was also proposed by Hain [22], who mentioned the graphical method of Tao and Bonnell [23,24] that can be considered a variation of the one proposed by Hall. A similar approach, but referred to the design of four-bar linkages with a prescribed maximum and minimum angular velocity ratios, based on the application of the Freudenstein theorem, is reported in Ref. [25]. Suh [26] extended the design of linkages to replace circular gears from the plane to the sphere and into the space, by using the displacement matrix method. Conversely, the problem of slider–crank mechanisms with quasi-constant transmission ratio was solved with a multiloop linkage and proposed in Ref. [27]. A more recent publication dealing with a similar topic is reported in Ref. [28], where both four-bar and slider–crank mechanisms were considered to generate a desired transmission ratio, but the authors do not refer to the particular case of the quasi-constant transmission ratio generation.

In this paper, a specific problem of function generators is investigated under different conditions (infinitesimal and finite displacements) and by means of the theories initiated by Burmester and Chebyshev. When properly designed, four-bar linkages or slider–crank mechanisms allow the transmission of motion between two parallel axes with a quasi-constant transmission ratio within a wide range of driving link rotation. Hence, as a prospective substitute of gears or pinion–rack pairs, the kinematic synthesis of quasi-constant transmission ratio four-bar linkages or slider–crank mechanisms, respectively, is relevant in modern machine design. This means that the centrodes of the input–output motion approximate the gear pitch circles or the pitch pair circle–straight line of gears and pinion–rack pairs, respectively. It is well known that the class of quasi-constant transmission ratio linkages should ensure a relationship between input and output motion parameters with least deviation from the linear one. In the case of a four-bar linkage, the input and output motions are those of the frame adjacent links. In a general planar four-bar linkage, excluding the particular case of parallelogram four-bar linkage, the constant transmission ratio condition cannot be achieved along the entire range of motion; for this reason, the term *quasi-constant* seems more appropriate. Similarly, as constant transmission ratio slider–crank linkage is herein intended the one with a linear relationship between the slider displacement and the crank or rocker rotation. Also, in this case, the requirement cannot be fulfilled within all the range of motion. In both cases, a meaningful index of performance is the structural error herein defined as the absolute difference between the generated transmission ratio and the required constant one [29]. It should be acknowledged that an optimal design depends on the optimum criterion adopted. In our case, we can state that:

— For the linkages (four-bar and slider–crank) whose design is based on infinitesimal Burmester theory, third-order accuracy is achieved. In other words, in a Taylor expansion, the mechanically generated function and the required one

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match first-, second-, and third-order derivatives. In the general case, constant transmission ratio with fourth-order accuracy cannot be achieved, as demonstrated, for the case of Cardan motion, by Bottema [30] and Freudenstein [31]. This means that, if we compare design alternatives on the basis of infinitesimal motions, the proposed procedures guarantee a design of four-bars and slider–cranks with the highest possible accuracy.

- For the linkages (four-bar and slider-crank) whose design is based on Chebyshev theory, assuming the so called min–max criterion for the structural error, the optimality of our solution is certified because all the conditions imposed satisfy those required for the optimum by the Chebyshev's theorem [32]. In other words, when the min–max criterion is adopted and the conditions required by Chebyshev's theorem are all satisfied, no further improvement is possible. In the mathematics domain, a classic outcome of the application of the min–max optimality criterion is the Chebyshev polynomials. This criterion was originally stated to solve a kinematic synthesis problem.
- For the above reasons, the authors prefer the development of design procedures based on Burmester and Chebyshev theories and avoided iterative procedures whose results could converge on a relative minimum instead of absolute minimum.

The following list should evidence our contribution:

- Extension of the Freudenstein's theorem to the synthesis of quasi-constant transmission ratio slider–crank.
- Kinematic procedure of quasi-constant transmission ratio for third-order accuracy. For the four-bar linkage, a graphical procedure has been proposed by Hall in his textbook. However, the one proposed in the paper is different. It is based on analytical equations and allows also transmission angle optimization. The case of slider–crank and the corresponding kinematic inversion are novel to the best of our knowledge.
- The extension of Chebyshev's min–max criterion for the design of quasi-constant transmission ratio slider–crank is also novel.
- The comparison of structural error for quasi-constant transmission linkages, whose design is based on infinitesimal Burmester theory and Chebyshev min–max criterion, is also new.

The paper is organized in two parts. The first one deals with the design of four-bar and slider–crank linkages by using infinitesimal Burmester theory. In particular, kinematic loci, such as inflection circle and cubic of stationary curvature, are used in order to ensure motion control up to third-order accuracy. Due to the constant transmission ratio requirement, as in cycloidal motion, the following conditions are simultaneously fulfilled: (a) the collineation axis is orthogonal to the coupler link, as stated by Freudenstein theorem and (b) the cubic of stationary curvature degenerates into a ϕ-curve [1]. These conditions allow a straightforward setup of design equations.

The second part focuses on the use of Chebyshev optimality criterion. To reduce algebraic burden, the synthesis of the four-bar and slider–crank linkages follows, respectively, from the one of a circular and straight path swinging-block linkage generators. For the cases of infinitesimal and finite displacements, the paper supplies ready-to-use design equations that guarantee the *best* solutions for both design theories implemented.

2 Kinematic Synthesis: Freudenstein Theorem

The kinematic synthesis of four-bar linkages and slider–crank mechanisms as function generators with quasi-constant transmission ratios τ can be carried out by means of the Freudenstein theorem, which states that the maximum and minimum values of τ are obtained for the mechanism configurations where the collineation axis is orthogonal to the coupler link. The application of the condition from Freudenstein theorem gives a quasi-constant transmission ratio four-bar linkage with second-order accuracy.

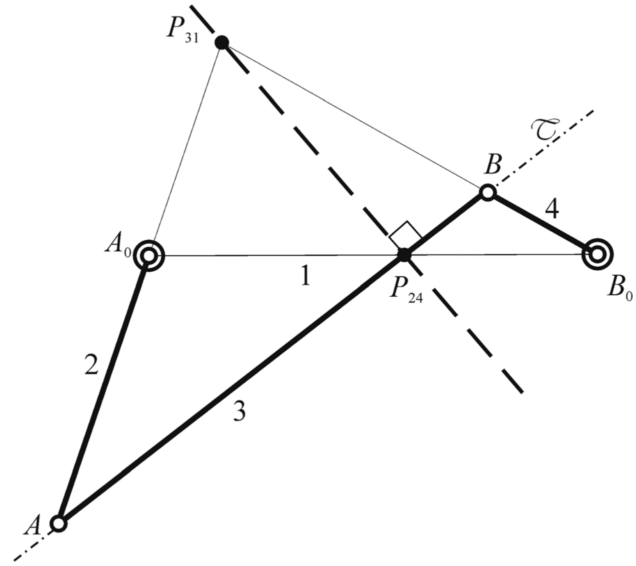

Fig. 1 Freudenstein theorem: four-bar linkage

In this case, in Fig. 1, the prescribed data are the center distance between the fixed revolute joints A_0 and B_0 and the prescribed transmission ratio $\tau = \omega_4/\omega_2$. The position of the instant center of rotation P_{24} for the relative motion between the driving link A_0A and the driven link B_0B is located in agreement with the Aronhold–Kennedy theorem. For simplicity, assuming the center distance A_0B_0 equal to unit value, the following equations can be established:

$$r = \frac{1}{1+\tau} \tag{1}$$

and

$$r' = \frac{\tau}{1+\tau} \tag{2}$$

where $r = A_0P_{24}$ and $r' = B_0P_{24}$.

The points A_0, B_0, and P_{24} are initially located consistently with Eqs. (1) and (2). Then, the coupler link direction must be established. For this purpose, a coupler line $\mathcal{C}$ passing through P_{24} is arbitrarily chosen along with the collineation axis perpendicular to $\mathcal{C}$ in P_{24}. Moreover, two arbitrarily chosen lines through A_0 and B_0 intersect $\mathcal{C}$ in A and B, respectively. The four-bar linkage with link lengths A_0A, AB, and A_0B generates a constant transmission ratio with second-order accuracy.

A similar approach can be extended to synthesize a slider–crank mechanism, whose transmission ratio τ is now defined as the ratio between the velocity of the point B and the angular velocity of the crank/rocker A_0A, as shown in Fig. 2. An approximate cycloidal motion between the driving crank/rocker link and the piston can be again generated through the application of Freudenstein theorem. In fact, when τ is generated with a second-order accuracy, the relative motion of A_0A with respect to the piston motion resembles the pure-rolling of a circle (moving centrode) on a straight line (fixed centrode).

Similarly, in the case of quasi-constant transmission ratio four-bar linkage, epicycloidal or hypocycloidal relative motion is approximated when the instant center of rotation P_{24} falls inside or outside the joining line across A_0 and B_0, respectively. Thus, when the piston axis and position of point B, along with the

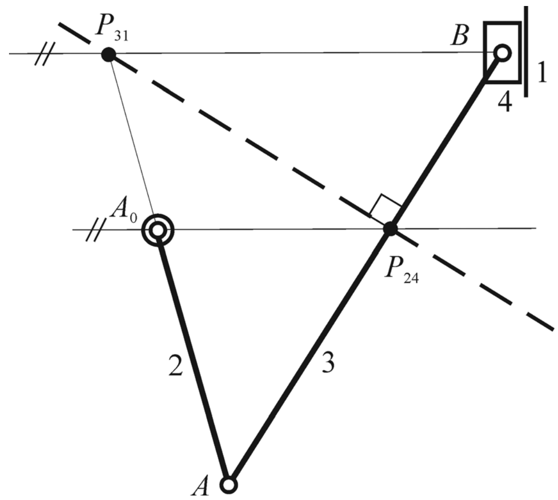

Fig. 2 Freudenstein theorem: slider–crank mechanism

position of the revolute joint A_0, are given, the kinematic synthesis of a second-order accuracy for a constant transmission ratio slider–crank mechanism can be carried out according to the following procedure:

(1) Trace the segment $A_0P_{24} = \tau$. Note that the piston axis will be necessarily orthogonal to this segment.
(2) Trace freely the segment A_0A representing the crank link;
(3) Trace the line joining A with P_{24}.
(4) Trace across P_{24} the perpendicular (collineation axis) to AP_{24} up to intersect in P_{31}, the extension of A_0A.
(5) Trace the line parallel to the segment A_0P_{24} from P_{31} up to intersect in B, the extension of AP_{24}.

Second-order accuracy is guaranteed by the fulfilment of the condition mentioned by Freudenstein theorem. In fact, the collineation axis is orthogonal to the coupler link, and, as expected, the piston axis is perpendicular to A_0P_{24}.

3 Kinematic Synthesis: Burmester Theory

In order to increase accuracy of the constant transmission ratio linkage up to the third order, the cubic of stationary curvature of the relative motion between driven and moving links must be taken into account.

In Fig. 3, the equation of the cubic of stationary curvature $\mathcal{C}$ can be conveniently expressed in polar form with respect to the canonical Cartesian frame $\mathcal{F}$ (X, Y), which shows the origin in the instantaneous center of rotation P_0, the X-axis (abscissa) tangent to both centrodes λ and l of radii of curvature r and r' in P_0, respectively, and the Y-axis (ordinate) passing through the center of the inflection circle $\mathcal{I}$ of diameter δ. The orientation of the Y-axis is such that δ is always positive. Thus, the equation of the cubic of stationary curvature can be expressed in the form

$$\frac{1}{h} = \frac{1}{M \sin\psi} + \frac{1}{N \cos\psi} \tag{3}$$

where

$$M = -\frac{3\delta}{\frac{\mathrm{d}\delta}{\mathrm{d}l}} \tag{4}$$

and

$$N = \frac{3rr'}{2r - r'} \tag{5}$$

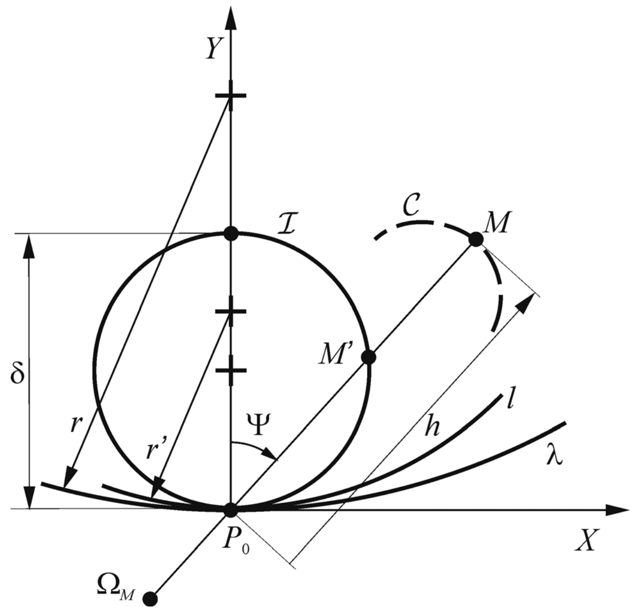

Fig. 3 Inflection circle and cubic of stationary curvature

are the motion invariants, h is the distance of point M from P_0, and ψ is the angle made between the Y-axis with the oriented segment P_0M.

3.1 Four-Bar Linkage. A velocity ratio τ can be expressed as ratio of the distances between the relative motion instantaneous centers. For instance, with reference to the four-bar linkage of Fig. 4, the transmission ratio between the two links adjacent to the frame is

$$\tau = \frac{\omega_2}{\omega_1} = \frac{P_0A_0}{P_0B_0} \tag{6}$$

In Fig. 4, if we let $a = P_0A_0$, then $P_0B_0 = (1 + a)$ and

$$\tau = \frac{a}{1 + a} = \frac{r'}{r} \tag{7}$$

When P_{24} is between A_0 and B_0, τ and a are negative quantities. In order to maintain this ratio stationary, the distances A_0P_0 and

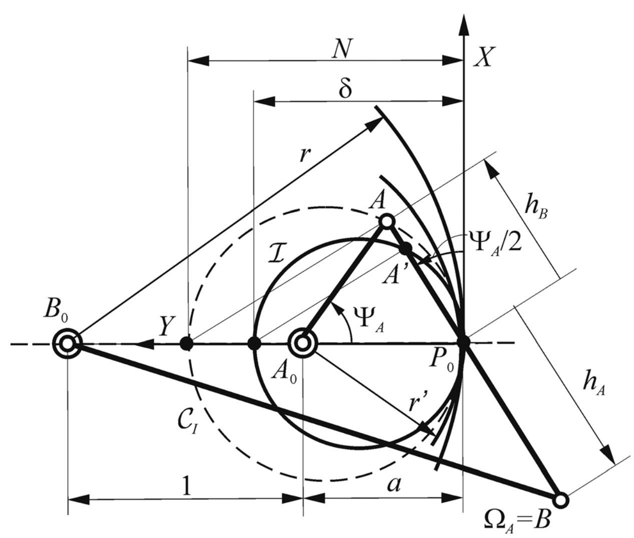

Fig. 4 Burmester theory: four-bar linkage

B_0P_0 must be also kept as much as possible constant. Since the velocity of displacement of P_0 along A_0B_0 is proportional to δ, to obtain a constant ratio for three infinitesimal displacements, $\mathrm{d}\delta/\mathrm{d}l = 0$ is required. Under this condition, the cubic of stationary curvature degenerates into two branches: the straight line $x = 0$ ($\mathcal{C}_{\mathrm{II}}$) and a circle $\mathcal{C}_{\mathrm{I}}$ through P_0 of diameter N and with center on the Y-axis. In particular, one has

$$N = \frac{3(1+a)a}{2+a} \tag{8}$$

In order to obtain a constant transmission ratio with third-order accuracy, the following conditions must be simultaneously fulfilled:

- The center of the revolute joint A belongs to the circle $\mathcal{C}_{\mathrm{I}}$.
- Point B must be the center of curvature of the trajectory of point A under the considered relative motion.

This last condition is achieved by applying the Euler–Savary equation. The computational flow is as follows:

(1) Choose angle ψ_A within the interval $(0,\pi)$ and compute the Cartesian coordinates x_A and y_A of point A from the knowledge of the polar coordinate h_A, as follows:

$$\begin{aligned} h_A &= N \sin\frac{\psi_A}{2} \\ x_A &= h_A \cos\frac{\psi_A}{2} \\ y_A &= h_A \sin\frac{\psi_A}{2} \end{aligned} \tag{9}$$

(2) Compute the inflection circle diameter δ from the Euler–Savary equation, as function of the radii of curvature r and r' of the centrodes, in the form

$$\delta = a(1+a) \tag{10}$$

(3) Compute the polar coordinate h_B of B by applying the second form of the Euler–Savary equation, as

$$h_B = h_A\left[1 - \frac{h_A}{h_A - \delta\sin(\psi_A/2)}\right] \tag{11}$$

(4) Compute the Cartesian coordinates of B, which is also the center of curvature Ω_A of point A, using

$$\begin{aligned} x_B &= h_B \cos\frac{\psi_A}{2} \\ y_B &= h_B \sin\frac{\psi_A}{2} \end{aligned} \tag{12}$$

Once the transmission ratio has been prescribed, there is only one free parameter, namely, ψ_A, that can be adjusted in order to obtain a four-bar linkage with feasible transmission angle and proportions.

This formulation allows the synthesis of a four-bar linkage with a third-order accuracy constant transmission ratio. The following numerical example validates this result.

If $\tau = 1/3$, $\psi_A = \pi/10$, and $a = 1/2$, one has: $N = 9/10$, $\delta = 3/4$, $x_A = 0.1373$, $y_A = 0.02145$, $x_B = -0.6864$, and $y_B = -0.1072$. The link length ratios are: $A_0A/A_0B_0 = 0.49785$; $AB/A_0B_0 = 0.83365$; and $B_0B/A_0B_0 = 1.74768$, as also reported in Fig. 5.

The synthesized four-bar linkage is shown in Fig. 5 along with the inflection circle and the circular branch of the cubic of stationary curvature. Since the collineation axis (dashed-line) is orthogonal to the coupler link AB and passes through the instant center P_0, the result is also consistent with Freudenstein theorem.

The transmission ratio τ is plotted in Fig. 6 versus the rotation angle θ of the driving link A_0A by giving a quasi-constant transmission ratio behavior for a wide angular range because of the third-order accuracy. In fact, for a driving link rotation of 90 deg, the deviation of the transmission ratio is about 2%.

3.2 Slider–Crank Mechanism. The kinematic synthesis of a slider–crank mechanism that approximates a cycloidal motion with a third-order accuracy can be carried out by applying the Burmester theory through the combined use of cubic of stationary curvature and inflection circle. In particular, this approach is quite convenient when the cubic degenerates into a ϕ-curve with two branches: a circle and a straight line passing through its center. In fact, in Fig. 7, the cubic of stationary curvature of Eq. (3) degenerates into the circle $\mathcal{C}_{\mathrm{I}}$ and the line $\mathcal{C}_{\mathrm{II}}$ coinciding with the Y-axis of the canonical Cartesian frame $\mathcal{F}$ (X, Y) for the relative motion between the piston 4 and the crank 2. Links 1 and 3 represent the fixed frame and the coupler link, respectively, as shown in Fig. 8. Thus, in order to approximate the required cycloidal motion between A_0A (crank 2) and the piston 4, the centrodes λ and l should match the straight line coinciding with the X-axis and the circle of center A_0 and radius r', respectively. The centrodes are tangent to each other at the relative instant center of rotation P_{24} of the slider-crank mechanism, which is labeled with P_0.

Because of the constancy for a cycloidal motion of the diameter δ of the inflection circle $\mathcal{I}$, the coefficient M of Eq. (4) approaches infinity. Consequently, Eq. (3) of the cubic of stationary curvature simplifies into

$$h = \frac{3}{2} r' \cos\psi \tag{13}$$

where ψ is the clockwise angle between the Y-axis and the polar coordinate h of $\mathcal{C}_{\mathrm{I}}$. Moreover, the radius of curvature r of the straight-line centrode λ approaches infinity and the coefficient N is equal to $1.5\, r'$, as deduced from Eq. (5).

Thus, Eq. (13) represents a circle of diameter N that is tangent to both centrodes in P_0, while the inflection circle I has a diameter δ equal to the radius of curvature of r' of the circular centrode l, because of the Euler–Savary equation and taking into account that $r \to \infty$. In this case, the transmission ratio τ can be expressed as

$$\tau = \frac{V_B}{\omega_1} = P_0A_0 \tag{14}$$

where V_B and ω_1 are the punctual velocity of point B and the angular velocity of the link A_0A, respectively.

In fact, for the relative cycloidal motion between 2 and 4, the velocity of point B is equal to $\omega_1\ P_0A_0$, as a pair of involute pinion–rack, where the action line coincides with the coupler link AB of the slider–crank mechanism, according to a third-order accuracy.

Therefore, a procedure for the kinematic synthesis of a slider–crank mechanism that approximates a cycloidal motion with a third-order accuracy is the following:

(1) Assume a pair of centrodes λ (straight line) and l (circle) of given radius r' for the relative motion between the crank 2 and the piston 4 of a slider–crank mechanism.
(2) Draw the circle branch $\mathcal{C}_{\mathrm{I}}$ of the cubic of stationary curvature, with diameter N given by Eq. (13).
(3) Draw the inflection circle $\mathcal{I}$ of diameter $\delta = r'$.
(4) Choose a point A on $\mathcal{C}_{\mathrm{I}}$ and determine its center of curvature Ω_A through the second form of the Euler–Savary equation by making use of the inflection circle by means of point A'.
(5) Assume Ω_A as coincident with point B of the coupler link AB of the designed slider–crank mechanism.

An alternative and equivalent procedure for the kinematic synthesis of a slider–crank approximating the relative cycloidal motion between crank and piston, with a third-order accuracy,

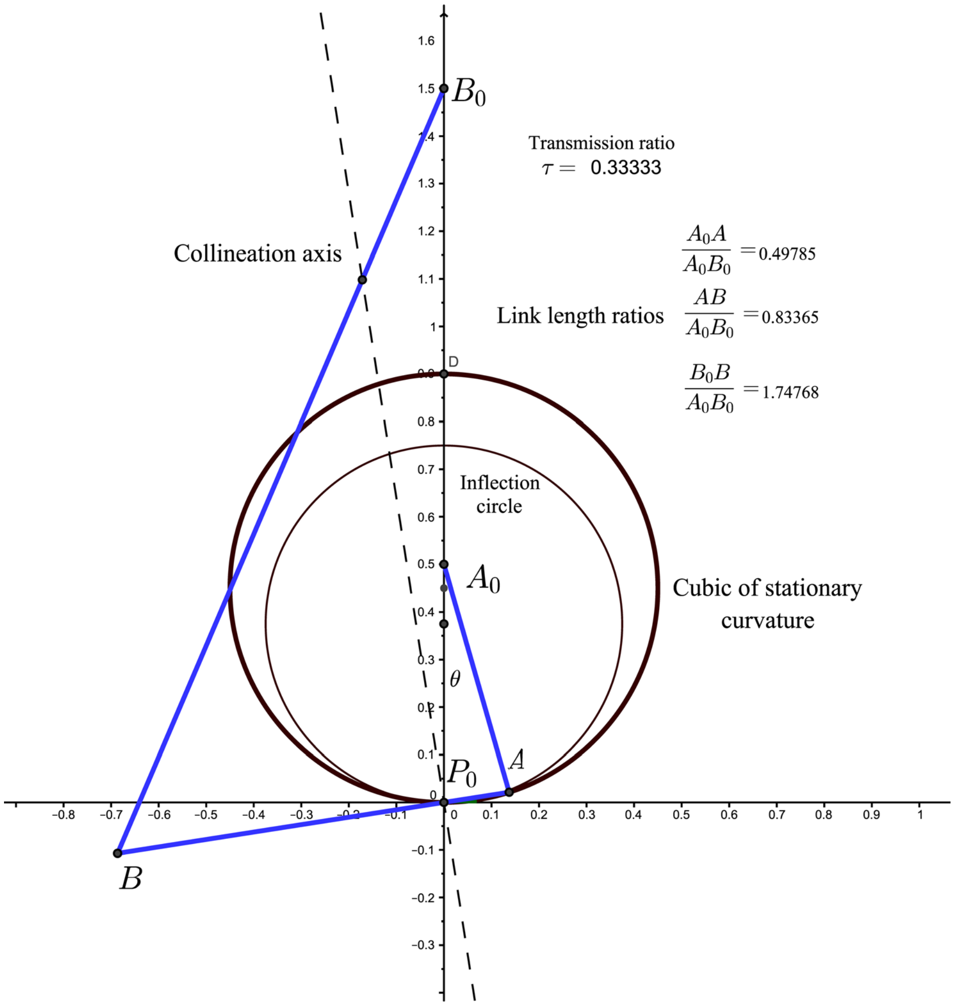

Fig. 5 Burmester theory: four-bar linkage

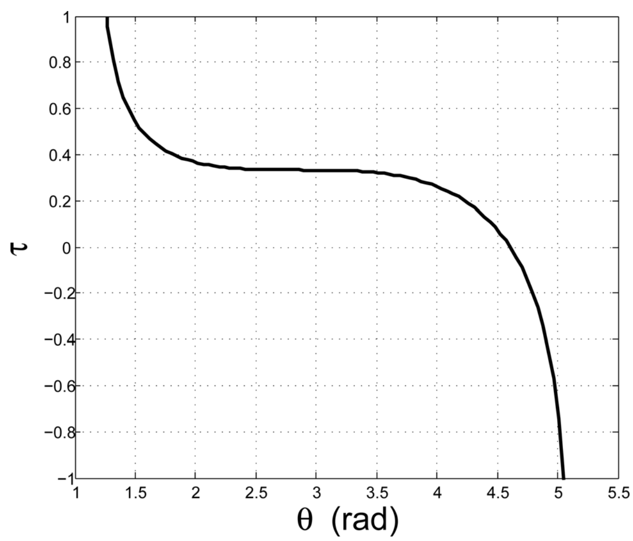

Fig. 6 Quasi-constant transmission ratio

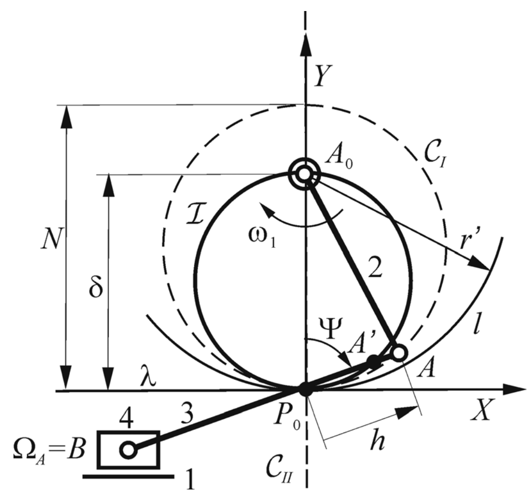

Fig. 7 Burmester theory: slider–crank mechanism

is based on the absolute coupler motion of a suitable auxiliary linkage $A_0\ A'B$, which is shown in dashed-line in Fig. 8. This mechanism has the coupler link $A'B$, along with the crank A_0A', equal and parallel to the crank 2 and the coupler 3, respectively, of the final slider–crank mechanism A_0AB, as shown in Fig. 8. In this way, the relative motion between the crank 2 and the coupler 3 can be conveniently analyzed by considering the absolute motion of the coupler link of the auxiliary mechanism $A_0\ A'B$, whose instant center of rotation is P^*_{24}, instead of P_{24} for the correspondent relative motion.

Thus, according to this alternative procedure, the kinematic synthesis of a slider–crank mechanism to approximate with a third-order accuracy, a relative cycloidal motion between the

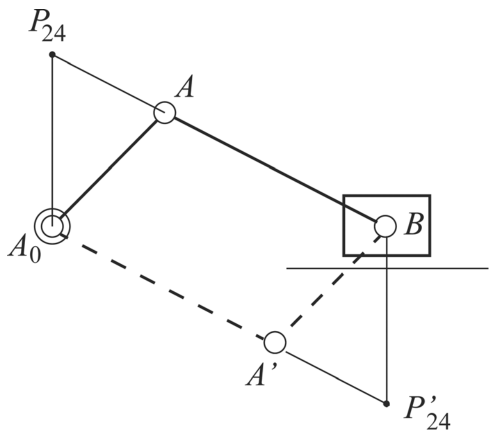

Fig. 8 Slider–crank mechanism (continuous-line) and its auxiliary mechanism (dashed-line)

crank 2 and piston 4 can be carried out by designing the auxiliary mechanism (dashed-line) through the same steps discussed above. Nevertheless, as soon as, the auxiliary mechanism has been designed by choosing freely A^* on the circle branch of the cubic of stationary curvature and determining A_0 through the application of the Euler–Savary equation, the correspondent slider–crank mechanism A_0AB can be easily obtained. The proposed formulation has been validated through several examples, as that shown in Fig. 9. The third-order accuracy guarantees an improvement with respect to the results produced by the simple application of the Freudenstein theorem. As a check of the results consistency, we observe that the collineation axis $P_{13}P_{24}$ is again orthogonal to the coupler link AB.

4 Kinematic Synthesis: Chebyshev Theory

The Chebyshev optimality criterion [32,33] is applied to obtain a compact closed-form solution for the dimensional synthesis of quasi-constant transmission ratio planar linkages.

As discussed above, the quasi-constant transmission ratio behavior of a four-bar linkage or slider–crank mechanism is aimed to approximate the motion of circular gears or pinion–rack, respectively. Pitch circles or circle and straight line correspond to the centrodes of the relative motion between driving and driven links or between crank and piston, respectively. However, the design specification of a quasi-constant transmission ratio or quasi-circular centrodes, circle and straight line in the case of slider–crank mechanism, can be also imposed by considering a swinging-block mechanism, as the one depicted in Fig. 10. The trajectory of coupler point M is required to approximate a circle or a straight line for the synthesis of a four-bar linkage or slider–crank mechanism, respectively. In fact, in Fig. 10, the swinging-block mechanism, which is an inversion of the slider–crank/rocker mechanism with the coupler link AP assumed as fixed frame, has a guide oscillating about the fixed revolute joint P and a coupler link AM with a plunging motion along the guide. This means that the kinematic synthesis of this mechanism can be developed with the aim to obtain a circular path generator, as in Fig. 10, or a straight-line path generator.

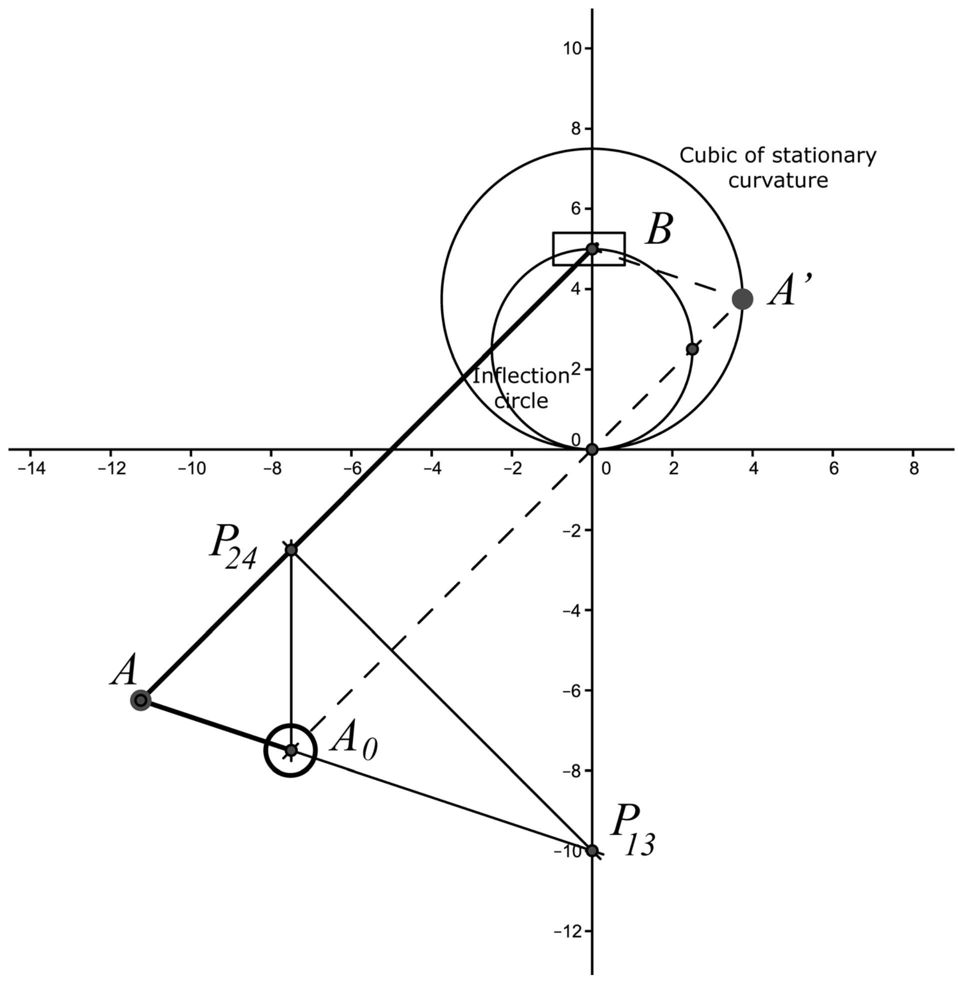

Fig. 9 Burmester theory: slider–crank mechanism

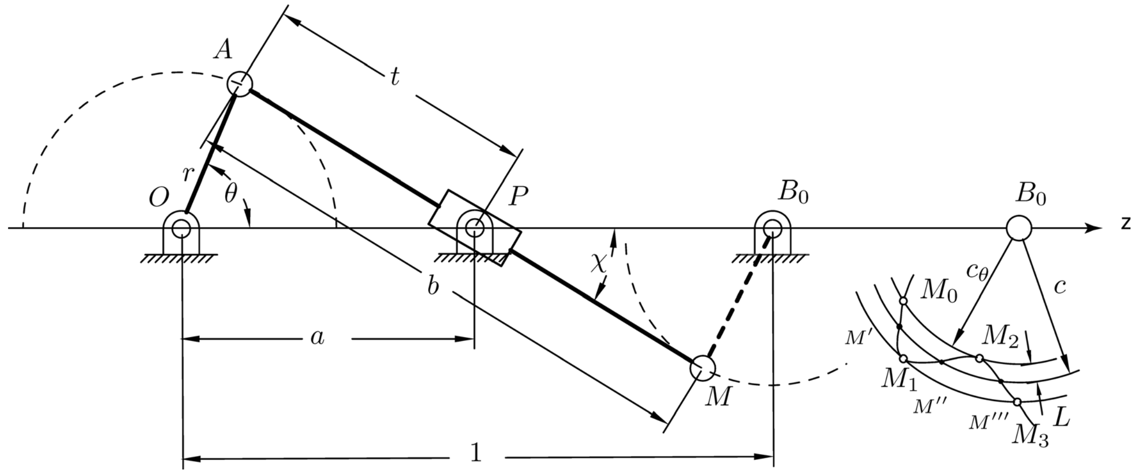

Fig. 10 Swinging-block circular path generator

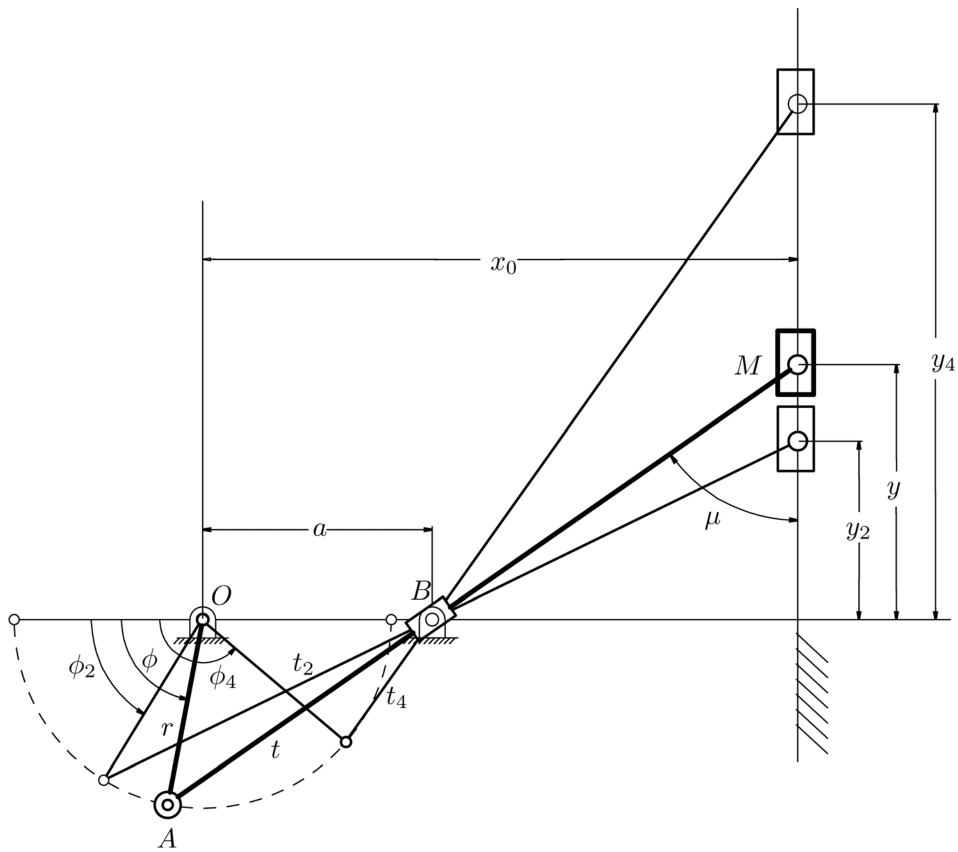

Fig. 11 Chebyshev theory: slider–crank mechanism

Thus, the design procedure initially considers a path generator swinging-block mechanism with the aim to remove the oscillating guide at point P and install a suitable link joining the coupler point M with its center of curvature B_0 through two revolute joints. Consequently, a quasi-constant transmission ratio between the driving link OA and the driven link B_0M is obtained. In fact, point P will play the role of the instant center of rotation P_0 for the relative motion between OA and B_0M. According to the Aronhold–Kennedy theorem O, $P = P_0$ and B_0 are aligned.

A similar approach can be extended to the synthesis of a slider–crank mechanism with an approximate relative cycloidal motion between crank and piston. In fact, the problem can be reduced to the kinematic synthesis of a swinging-block straight-line path generator. The oscillating guide in P is removed and a pin-in-the-slot pair is added at the coupler point M whose trajectory approximates a straight line. In Fig. 11, the pin-in-the-slot pair is herein sketched as combination of revolute and prismatic joints. Moreover, it is noteworthy that a theoretical constant transmission ratio behavior of function generators four-bar linkage, or a slider–crank mechanism, would be obtained when $P = P_0$ remains fixed with the frame, as in the case of circular gears or pinion–rack (cycloidal motions). A constant transmission ratio behavior within the entire working range is not in general possible with these linkages and, thus, the kinematic synthesis is aimed to obtain a quasi-constant transmission ratio. Freudenstein theorem and the Burmester theory guarantee, respectively, second-order and third-order accuracy for infinitesimally separated positions.

Considering finite displacements, the Chebyshev optimality criterion is applied to formulate the kinematic synthesis of swinging-block circular or straight-line path generators with the aim to obtain a four-bar linkage and a slider–crank mechanism, respectively. In the geometry and nomenclature of Fig. 10, the coupler point M is chosen such that an approximate circular path of center B_0 on the X-axis is traced. This choice allows to take advantage of the linkage symmetry and to increase the number of precision points. The radius c of the path is not prescribed and the distance OB_0 is assumed to be equal to unit value. Thus, in order to avoid square roots, the structural error is defined as follows:

$$\varepsilon = c_\theta^2 - c^2 \tag{15}$$

where c_θ is the distance B_0M, and c is the required link length of the four-bar linkage $OAMB_0$.

In order to express the structural error ε as function of design parameters a, b, and r, considering both triangles B_0MP and OAP, one has

$$\begin{aligned} c_\theta^2 &= (1-a)^2 + (b-t)^2 - 2(b-t)(1-a)\cos\chi \\ r^2 &= a^2 + t^2 - 2at\cos\chi \end{aligned} \tag{16}$$

Eliminating $\cos\chi$ from Eq. (16), one has

$$c_\theta^2 = \frac{t^2}{a} - b\frac{1+a}{a}t + b^2 + 1 + r^2 - a - \frac{r^2}{a} - \frac{b(a^2-r^2)(1-a)}{at} \tag{17}$$

Consequently, Eq. (15) can be expressed in polynomial form

$$\varepsilon = A\left(t^2 + p_1 t + p_2 + p_3\frac{1}{t}\right) \tag{18}$$

where

$$A = \frac{1}{a} \tag{19}$$

$$p_1 = -b(1+a) \tag{20}$$

$$p_2 = a\left(b^2 + 1 + r^2 - c^2 - a - \frac{r^2}{a}\right) \tag{21}$$

$$p_3 = -b\left(a^2 - r^2\right)(1-a) \tag{22}$$

The coefficients of the polynomial function of Eq. (18) must be computed in such a way to obtain the typical equi-oscillation behavior for the structural error curve. t_0, t_1, t_2, and t_3 are denoted the values of variable length t, where the structural error curve attains extreme values.

The Chebyshev optimality criterion will be applied to the second degree polynomial $P(t) = a\ \varepsilon$ of Eq. (18). Considering three precision points, one has three parameters p_1, p_2, and p_3 of Eq. (18) as function of t_0, t_1, t_2, and t_3 in the form

$$p_1 = -(2t_1 - t_3) = -(t_0 + 2t_2) \tag{23}$$

$$p_2 - aL_q = t_1^2 + 2t_1t_3 \tag{24}$$

$$p_2 + aL_q = 2t_0t_2 + t_2^2 \tag{25}$$

$$p_3 = -t_1^2 t_3 = -t_0 t_2^2 \tag{26}$$

From Eqs. (21) and (22), one has

$$p_2 = \frac{t_1^2 + t_2^2 + 2t_0t_2 + 2t_1t_3}{2} \tag{27}$$

which is substituted in Eq. (21) to express the nominal radius c of the approximated circular path of point M as follows:

$$c = \sqrt{b^2 + 1 + r^2 - a - \frac{r^2}{a} - \frac{t_1^2 + t_2^2 + 2t_0t_2 + 2t_1t_3}{2a}} \tag{28}$$

Moreover, from Eqs. (24) and (25), one has

$$L_q = \frac{1}{2a}\left(t_2^2 + 2t_0t_2 - t_1^2 - 2t_1t_3\right) \tag{29}$$

After, we let

$$L = c_\theta - c \tag{30}$$

the structural error can be approximately evaluated in adimensional form by means of the following equation:

$$L \simeq \frac{L_q}{2c} \tag{31}$$

Moreover, the extreme angular positions θ_0 and θ_3 of the crank angle within the working range can be, respectively, expressed in the form

$$\cos\theta_0 = \frac{r^2 + a^2 - t_0^2}{2ra} \tag{32}$$

$$\cos\theta_3 = \frac{r^2 + a^2 - t_3^2}{2ra} \tag{33}$$

by referring to the triangle OAP for the extreme values t_0 and t_3 of the parameter t.

Substituting Eqs. (20) and (22) into Eq. (23), with the aim to obtain the length b of the coupler link AM, one has

$$b = \frac{t_0^2(1+a) + 2(a^2-r^2)(1-a)}{t_0(1+a)^2} \pm \frac{\sqrt{t_0^2(a^2-r^2)(1-a^2) + (a^2-r^2)^2(1-a)^2}}{t_0(1+a)^2} \tag{34}$$

The sign before the square root must be chosen such that b is greater than t_0 with

$$a - r \le t_0 \le a + r \tag{35}$$

Since t_0 is usually prescribed, t_2 can be expressed as function of t_0 as follows:

$$t_2 = \frac{b(1+a) - t_0}{2} \tag{36}$$

by equating the second terms of Eqs. (20) and (23). Likewise, t_3 and t_1 are given by

$$t_3 = \frac{1}{2}\left(4t_2 + t_0 \pm \sqrt{t_0^2 + 8t_0t_2}\right) \tag{37}$$

$$t_1 = \frac{1}{2}\left(t_0 \pm \sqrt{t_0 t_3}\right) \tag{38}$$

Therefore, the computational steps for the kinematic synthesis of swinging-block circular path generators by means of the Chebyshev min–max criterion with the final target to design a quasi-constant transmission ratio four-bar linkage are the followings:

(1) Compute b by means of Eq. (34) with the requirement that $b > t_0$.

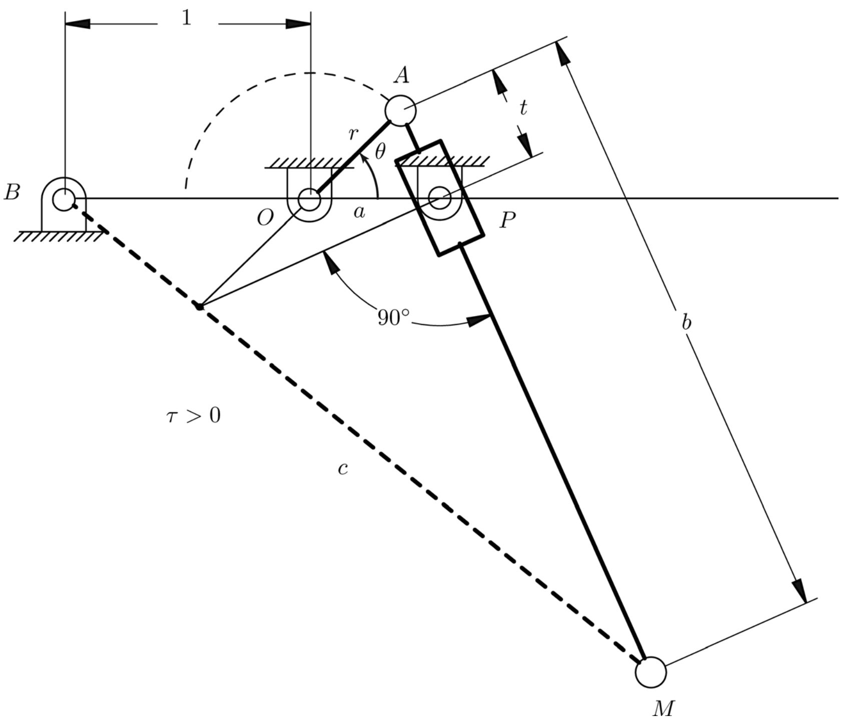

Fig. 12 Chebyshev theory: four-bar linkage for $\tau>0$, $a<1$, and $t_0>0$

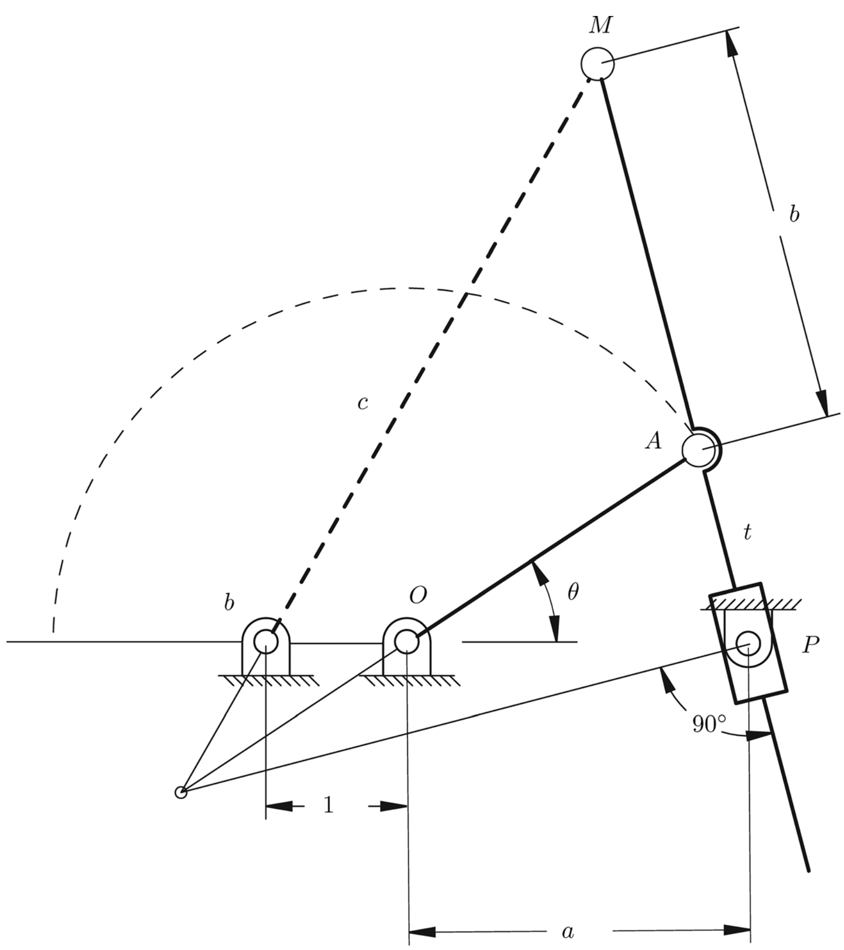

Fig. 13 Chebyshev theory: four-bar linkage for $\tau>0$, $a>1$, and $t_0<0$

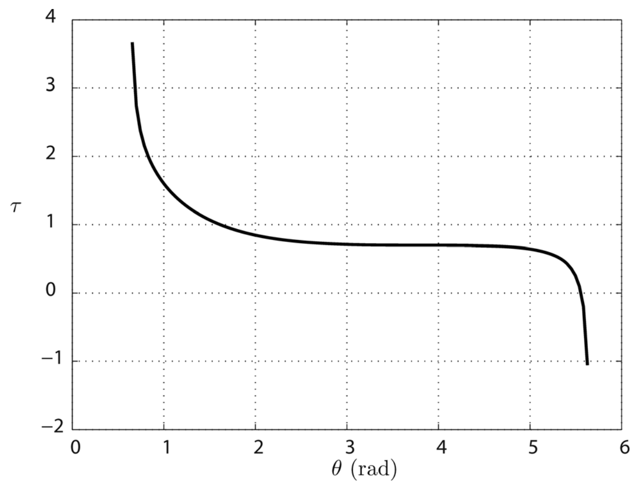

Fig. 14 Four-bar linkage: quasi-constant transmission ratio

(2) Compute t_2, t_3, and t_1 by means of Eqs. (36)–(38), respectively, with the requirement that $t_0 > t_1 > t_2 > t_3$.
(3) Compute c by using Eq. (28).
(4) Compute θ_0 and θ_3 through Eqs. (32) and (33), respectively, and the crank rotation $\theta_m = \theta_3 - \theta_0$.
(5) Compute the structural error L by means of Eq. (31).

The input data of the proposed algorithm are a, r, and t_0, while the output are c, b, θ_0, θ_3, L, t_0, t_1, t_2, and t_3.

In Fig. 10, the transmission ratio τ depends by the position of point P and it is given by

$$\tau = \frac{a}{1-a} \tag{39}$$

which is negative, when P is located in between the segment OB_0 (external spur gears) and positive when it falls outside (internal spur gears), as in the examples of Figs. 12 and 13.

Figure 12 corresponds to the case of $\tau > 0$, $a < 1$ and $t_0 > 0$, while the example of Fig. 13 refers to the case of $\tau > 0$, $a > 1$, and $t_0 < 0$. Since the transmission ratio τ depends on the position of point P, which should coincide with the relative instant center of rotation P_0 of the final four-bar linkage $OAMB_0$, it is required that, for a given range of motion, this point maintains as long as possible a given position. This will ensure a quasi-constant transmission ratio behavior.

Figures 10, 12, and 13 show the swinging-block mechanism OAP with an oscillating guide at the fixed point P. When the coupler point M approximates a circular arc with center B_0, within a suitable range of motion between two extreme values of the crank angle θ, this mechanism can be transformed into a quasi-constant transmission ratio four-bar linkage. In particular, $a = 2.3333$ for $\tau = 0.7$ and assuming $r = 2.4$ and $t_0 = -1.5$, one has $b = 3.4622$ and $c = 5.1568$. The diagram of the transmission ratio τ versus the crank angle θ is shown in Fig. 14.

The constancy of τ in a wide angular range can be appreciated in agreement with the required quasi-constant transmission ratio behavior.

Moreover, it is interesting to consider the particular case for $t_0 = t_1 = t_2 = t_3$, which means to have the three precision points infinitesimally close to each other, as in the case which has been considered through the application of the Burmester theory. In fact, t_0, t_1, t_2, and t_3 are usually spaced among them by giving three different precision points for three spaced positions of the mechanism. Thus, the proposed formulation takes the form

$$t_0 = \sqrt{\frac{3(a^2 - r^2)(1-a)}{1+a}} \tag{40}$$

and, likewise

$$b = \frac{3}{1+a} t_0 \tag{41}$$

$$c = \sqrt{b^2 + r^2 + 1 - a - \frac{1}{a}(r^2 - 3t_0^2)} \tag{42}$$

The initial angular position θ_0 of the driving link A_0A is

$$\theta_0 = \cos^{-1}\left(\frac{r^2 + a^2 - t_0^2}{2ar}\right) \tag{43}$$

A similar approach can be applied to the kinematic synthesis of a slider–crank mechanism, which approximates a relative cycloidal motion between the crank and the piston, as developed in Sec. 3.2 Slider-Crank Mechanism through the application of the Burmester theory. A swinging-block straight-line path generator can be conveniently considered to design such a slider–crank mechanism. In the geometry and nomenclature of Fig. 15, the coupler point M should approximate a straight line for three finite displacements of the mechanism.

After the swinging-block mechanism is designed applying the Chebyshev optimality criterion, the oscillating guide at point B is removed and a suitable piston is added and joined through a revolute joint at the coupler point M.

Therefore, the approach is similar to that described and applied for the synthesis of a quasi-constant transmission ratio four-bar linkage. In the geometry of Fig. 11, the slider velocity V_M can be expressed as follows:

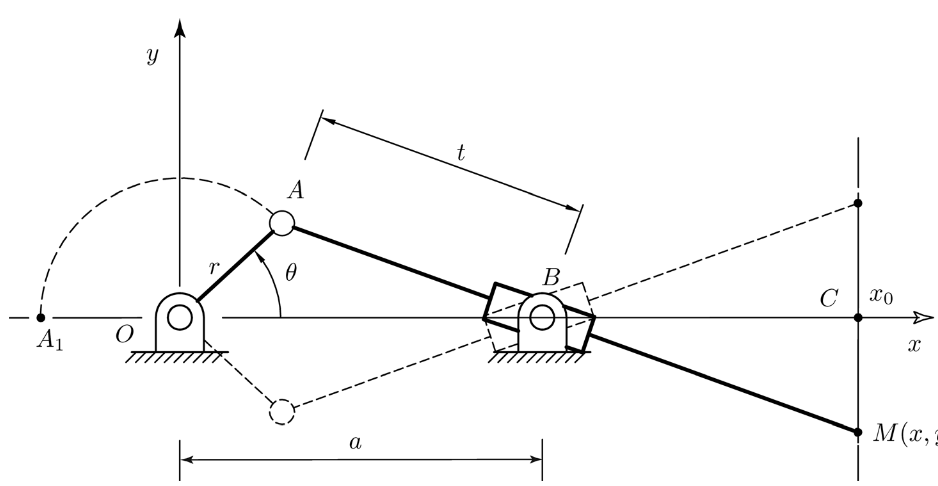

Fig. 15 Swinging-block straight path generator

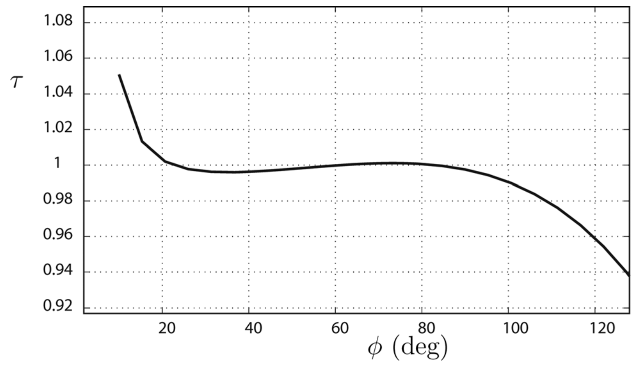

Fig. 16 Slider–crank mechanism: quasi-constant transmission ratio

$$V_M = \omega_1 OB \quad (44)$$

where ω_1 is the angular velocity of the crank OA, and OB is the frame length a of the swinging-block mechanism.

Hence, in order to obtain a quasi-constant transmission ratio or an approximated cycloidal motion between the crank OA and the piston in M, the slider velocity V_B should be kept constant and this is possible by maintaining constant the distance $OB = a$. For this purpose, a suitable swinging-block straight line path generator is associated to the slider-crank mechanism with minimized displacement of point B along the abscissa axis.

Thus, the swinging-block is designed to generate a straight line along the slider axis. This type of synthesis carried out by means of the Chebyshev criterion is outlined in different sources (e.g., Refs. [34–36]) and will be not herein repeated.

In this analysis, it is assumed $a = OB = \tau$, where τ is the ratio between V_B and ω_1. The crank length r has a unit length. The following numerical example has been carried out in order to validate the proposed formulation. Let $\tau = 1$ the velocity ratio to be generated and $r = 1$ the crank length. The following results have been obtained: $b = 5.9495$, $t_1 = 2.5$, $t_2 = 2.2208$, $t_3 = 1.7247$, $t_4 = 1.5079$, and $x_0 = 4.9546$. The diagram of τ versus the crank angle ϕ is shown in Fig. 16, where a satisfactory constancy of τ about the unit value can be appreciated for a wide range of the crank angle.

The algorithm for the kinematic synthesis of slider–crank mechanisms to approximate a relative cycloidal motion between crank and piston, which refers to the sketches of Figs. 11 and 15, is not reported here, but its results are conveniently summarized in the design chart of Fig. 17.

In fact, the ratios (b/a), (x_0/a), and (d/a) are plotted as function of the ratio (r/a) between the crank length r and the distance $a = AB$. In particular, d is the stroke of the coupler point M in which the straight line is approximated and, consequently, d is also the stroke of the piston. Therefore, a slider–crank mechanism that obeys the required design specification can be designed by assigning r and a, and determining d, b, and x_0 from Fig. 17.

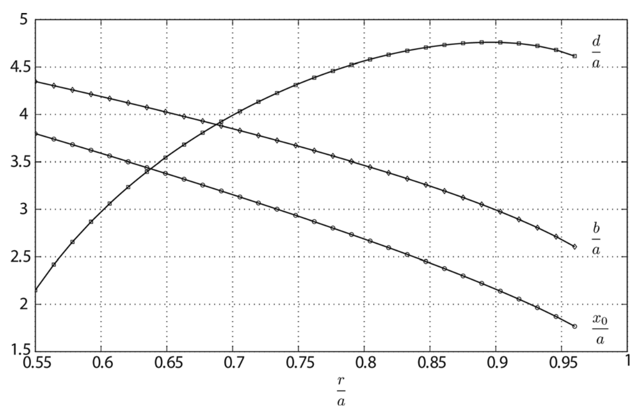

Fig. 17 Design chart for slider–crank mechanisms with quasi-constant transmission ratio

Entering the design chart of Fig. 17 with a prescribed value of x_0/a, the remaining proportions of the slider–crank can be estimated accordingly. This gives the possibility to design a quasi-constant transmission ratio slider–crank by prescribing the length of the working range.

5 Conclusions

The paper discussed different methodologies for the kinematic synthesis of four-bar linkages and slider–crank mechanisms with quasi-constant transmission ratio. The first method is based on the application of Burmester theory for two and three infinitesimally separated positions in order to achieve second- and third-order accuracy, respectively. Then, the Chebyshev criterion is applied toward the synthesis of the class of linkages under consideration. Observing the structural error plots, one can conclude that:

— the accuracy of the mechanism designed through the Burmester method is high at the precision point and acceptable within a limited range of motion
— the Chebyshev method, which is based on finite displacements, offers a wider range of approximation but a lower overall accuracy.

All the discussed methods have been validated by means of numerical examples and plots of the transmission ratio.

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