Equations of motion of a double pendulum by means of the Principle of Virtual Work

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(Angles theta1 and theta 2 are measured w.r.t. the vertical line)

- L1, L2: Lengths of the two bars

- m1, m2: masses

- Ig1, Ig2: Moments of inertia with respect to mass centers

- g : acceleration gravity

Define absolute coordinates of centers of mass G1 and G2

```
(%i20) xG1:L1*sin(theta1(t))/2;
       yG1:-L1*cos(theta1(t))/2;
       xG2:L1*sin(theta1(t))+L2*sin(theta2(t))/2;
       yG2:-L1*cos(theta1(t))-L2*cos(theta2(t))/2;
```

$$(\%o20)\quad \frac{\sin\left(\text{theta1}\left(t\right)\right)\ L1}{2}$$

$$(\%o21)\quad -\frac{\cos\left(\text{theta1}\left(t\right)\right)\ L1}{2}$$

$$(\%o22)\quad \frac{\sin\left(\text{theta2}\left(t\right)\right)\ L2}{2}+\sin\left(\text{theta1}\left(t\right)\right)\ L1$$

$$(\%o23)\quad -\frac{\cos\left(\text{theta2}\left(t\right)\right)\ L2}{2}-\cos\left(\text{theta1}\left(t\right)\right)\ L1$$

```
(%i24) dxG1:diff(xG1,theta1(t))*dtheta1;
       dyG1:diff(yG1,theta1(t))*dtheta1;
       dxG2:diff(xG2,theta1(t))*dtheta1+diff(xG2,theta2(t))*dtheta2;
       dyG2:diff(yG2,theta1(t))*dtheta1+diff(yG2,theta2(t))*dtheta2;
```

$$(\%o24)\quad \frac{dtheta1\cos\left(\text{theta1}\left(t\right)\right)\ L1}{2}$$

$$(\%o25)\quad \frac{dtheta1\sin\left(\text{theta1}\left(t\right)\right)\ L1}{2}$$

$$(\%o26)\quad \frac{dtheta2\cos\left(\text{theta2}\left(t\right)\right)\ L2}{2}+dtheta1\cos\left(\text{theta1}\left(t\right)\right)\ L1$$

$$(\%o27)\quad \frac{dtheta2\sin\left(\text{theta2}\left(t\right)\right)\ L2}{2}+dtheta1\sin\left(\text{theta1}\left(t\right)\right)\ L1$$

Form the vectors dG1 and dG2 (Virtual displacements of centers of mass G1 and G2, respectively)

```
(%i28) dG1:matrix([dxG1],[dyG1]);
       dG2:matrix([dxG2],[dyG2]);
```

$$(\%o28)\quad \begin{pmatrix} \frac{dtheta1\cos(\text{theta1}(t))\,L1}{2} \\ \frac{dtheta1\sin(\text{theta1}(t))\,L1}{2} \end{pmatrix}$$

$$(\%o29)\quad \begin{pmatrix} \frac{dtheta2\cos(\text{theta2}(t))\,L2}{2}+dtheta1\cos\left(\text{theta1}\left(t\right)\right)\,L1 \\ \frac{dtheta2\sin(\text{theta2}(t))\,L2}{2}+dtheta1\sin\left(\text{theta1}\left(t\right)\right)\,L1 \end{pmatrix}$$

Compute virtual works of all external generalized forces (inertia forces included)

```
(%i30) dWi:-m1*diff(xG1,t,2)*dxG1-m1*diff(yG1,t,2)*dyG1-m2*diff(xG2,t,2)*dxG2-m2*diff(yG2,t,
       Ig1*diff(theta1(t),t,2)*dtheta1-Ig2*diff(theta2(t),t,2)*dtheta2-m1*g*dyG1-m2*g*dyG2;
```

$$(\%o30)\quad -m2\left(\frac{dtheta2\sin\left(\text{theta2}\left(t\right)\right)\,L2}{2}+dtheta1\sin\left(\text{theta1}\left(t\right)\right)\,L1\right)\left(\frac{\sin\left(\text{theta2}\left(t\right)\right)\left(\frac{d^2}{d\,t^2}\,\text{theta2}\left(t\right)\right)\,L2}{2}+\right.$$
$$\left.\frac{\cos\left(\text{theta2}\left(t\right)\right)\left(\frac{d}{d\,t}\,\text{theta2}\left(t\right)\right)^2\,L2}{2}+\sin\left(\text{theta1}\left(t\right)\right)\left(\frac{d^2}{d\,t^2}\,\text{theta1}\left(t\right)\right)\,L1+\cos\left(\text{theta1}\left(t\right)\right)\left(\frac{d}{d\,t}\,\text{theta1}\left(t\right)\right)^2\,L1\right)$$
$$m2\left(\frac{dtheta2\cos\left(\text{theta2}\left(t\right)\right)\,L2}{2}+dtheta1\cos\left(\text{theta1}\left(t\right)\right)\,L1\right)\left(\frac{\cos\left(\text{theta2}\left(t\right)\right)\left(\frac{d^2}{d\,t^2}\,\text{theta2}\left(t\right)\right)\,L2}{2}-\right.$$
$$\left.\frac{\sin\left(\text{theta2}\left(t\right)\right)\left(\frac{d}{d\,t}\,\text{theta2}\left(t\right)\right)^2\,L2}{2}+\cos\left(\text{theta1}\left(t\right)\right)\left(\frac{d^2}{d\,t^2}\,\text{theta1}\left(t\right)\right)\,L1-\sin\left(\text{theta1}\left(t\right)\right)\left(\frac{d}{d\,t}\,\text{theta1}\left(t\right)\right)^2\,L1\right)$$
$$g\,m2\left(\frac{dtheta2\sin\left(\text{theta2}\left(t\right)\right)\,L2}{2}+dtheta1\sin\left(\text{theta1}\left(t\right)\right)\,L1\right)-\frac{dtheta1\,m1\sin\left(\text{theta1}\left(t\right)\right)\,L1\left(\frac{\sin(\text{theta1}(t))\left(\right.\right.}{}$$
$$\frac{dtheta1\,m1\cos\left(\text{theta1}\left(t\right)\right)\,L1\left(\frac{\cos(\text{theta1}(t))\left(\frac{d^2}{d\,t^2}\,\text{theta1}(t)\right)L1}{2}-\frac{\sin(\text{theta1}(t))\left(\frac{d}{d\,t}\,\text{theta1}(t)\right)^2L1}{2}\right)}{2}-$$
$$\frac{dtheta1\,g\,m1\sin\left(\text{theta1}\left(t\right)\right)\,L1}{2}-dtheta2\,Ig2\left(\frac{d^2}{d\,t^2}\,\text{theta2}\left(t\right)\right)-dtheta1\,Ig1\left(\frac{d^2}{d\,t^2}\,\text{theta1}\left(t\right)\right)$$

Separate factors of dtheta1 and dtheta2

```
(%i31) eq1:expand(ratcoef(dWi,dtheta1));
       eq2:expand(ratcoef(dWi,dtheta2));
```

$$(\%o31)\quad -\frac{m2\sin\left(\text{theta1}\left(t\right)\right)\sin\left(\text{theta2}\left(t\right)\right)\left(\frac{d^2}{d\,t^2}\,\text{theta2}\left(t\right)\right)\,L1\,L2}{2}-\frac{m2\cos\left(\text{theta1}\left(t\right)\right)\cos\left(\text{theta2}\left(t\right)\right)\left(\frac{d^2}{d\,t^2}\,\text{th}\right.}{}$$
$$\frac{m2\cos\left(\text{theta1}\left(t\right)\right)\sin\left(\text{theta2}\left(t\right)\right)\left(\frac{d}{d\,t}\,\text{theta2}\left(t\right)\right)^2\,L1\,L2}{2}-\frac{m2\sin\left(\text{theta1}\left(t\right)\right)\cos\left(\text{theta2}\left(t\right)\right)\left(\frac{d}{d\,t}\,\text{theta2}\left(t\right)\right)^2\,L1}{2}$$

$$m2\sin\left(\text{theta1}\left(t\right)\right)^2\left(\frac{d^2}{d\,t^2}\,\text{theta1}\left(t\right)\right)L1^2-\frac{m1\sin\left(\text{theta1}\left(t\right)\right)^2\left(\frac{d^2}{d\,t^2}\,\text{theta1}\left(t\right)\right)L1^2}{4}-$$
$$m2\cos\left(\text{theta1}\left(t\right)\right)^2\left(\frac{d^2}{d\,t^2}\,\text{theta1}\left(t\right)\right)L1^2-\frac{m1\cos\left(\text{theta1}\left(t\right)\right)^2\left(\frac{d^2}{d\,t^2}\,\text{theta1}\left(t\right)\right)L1^2}{4}-$$
$$g\,m2\sin\left(\text{theta1}\left(t\right)\right)\,L1-\frac{g\,m1\sin\left(\text{theta1}\left(t\right)\right)\,L1}{2}-Ig1\left(\frac{d^2}{d\,t^2}\,\text{theta1}\left(t\right)\right)$$

(%o32)
$$-\frac{m2\sin\left(\text{theta2}\left(t\right)\right)^2\left(\frac{d^2}{d\,t^2}\,\text{theta2}\left(t\right)\right)L2^2}{4}-\frac{m2\cos\left(\text{theta2}\left(t\right)\right)^2\left(\frac{d^2}{d\,t^2}\,\text{theta2}\left(t\right)\right)L2^2}{4}-$$
$$\frac{m2\sin\left(\text{theta1}\left(t\right)\right)\left(\frac{d^2}{d\,t^2}\,\text{theta1}\left(t\right)\right)\sin\left(\text{theta2}\left(t\right)\right)\,L1\,L2}{2}-\frac{m2\cos\left(\text{theta1}\left(t\right)\right)\left(\frac{d}{d\,t}\,\text{theta1}\left(t\right)\right)^2\sin\left(\text{theta2}\left(t\right)\right)\,L}{2}$$
$$\frac{m2\cos\left(\text{theta1}\left(t\right)\right)\left(\frac{d^2}{d\,t^2}\,\text{theta1}\left(t\right)\right)\cos\left(\text{theta2}\left(t\right)\right)\,L1\,L2}{2}+\frac{m2\sin\left(\text{theta1}\left(t\right)\right)\left(\frac{d}{d\,t}\,\text{theta1}\left(t\right)\right)^2\cos\left(\text{theta2}\left(t\right)\right)\,L}{2}$$
$$\frac{g\,m2\sin\left(\text{theta2}\left(t\right)\right)\,L2}{2}-Ig2\left(\frac{d^2}{d\,t^2}\,\text{theta2}\left(t\right)\right)$$

Collect terms

```
(%i33) eq1:facsum(eq1,diff(theta2(t),t,2),diff(theta1(t),t,2),diff(theta1(t),t),diff(theta2
       eq2:facsum(eq2,diff(theta2(t),t,2),diff(theta1(t),t,2),diff(theta1(t),t),diff(theta2
```

(%o33)
$$\left(-2\,m2\,\left(\sin\left(\text{theta1}\left(t\right)\right)\sin\left(\text{theta2}\left(t\right)\right)+\cos\left(\text{theta1}\left(t\right)\right)\cos\left(\text{theta2}\left(t\right)\right)\right)\left(\frac{d^2}{d\,t^2}\,\text{theta2}\left(t\right)\right)L1\,L2+\right.$$
$$2\,m2\,\left(\cos\left(\text{theta1}\left(t\right)\right)\sin\left(\text{theta2}\left(t\right)\right)-\sin\left(\text{theta1}\left(t\right)\right)\cos\left(\text{theta2}\left(t\right)\right)\right)\left(\frac{d}{d\,t}\,\text{theta2}\left(t\right)\right)^2L1\,L2-$$
$$\left(\frac{d^2}{d\,t^2}\,\text{theta1}\left(t\right)\right)\left(4\,m2\sin\left(\text{theta1}\left(t\right)\right)^2L1^2+m1\sin\left(\text{theta1}\left(t\right)\right)^2L1^2+4\,m2\cos\left(\text{theta1}\left(t\right)\right)^2L1^2+\right.$$
$$\left.m1\cos\left(\text{theta1}\left(t\right)\right)^2L1^2+4\,Ig1\right)-2\,g\,\left(2\,m2+m1\right)\sin\left(\text{theta1}\left(t\right)\right)\,L1\Big)/4$$

(%o34)
$$\left(-\left(\frac{d^2}{d\,t^2}\,\text{theta2}\left(t\right)\right)\left(m2\sin\left(\text{theta2}\left(t\right)\right)^2L2^2+m2\cos\left(\text{theta2}\left(t\right)\right)^2L2^2+4\,Ig2\right)-\right.$$
$$2\,m2\left(\frac{d^2}{d\,t^2}\,\text{theta1}\left(t\right)\right)\left(\sin\left(\text{theta1}\left(t\right)\right)\sin\left(\text{theta2}\left(t\right)\right)+\cos\left(\text{theta1}\left(t\right)\right)\cos\left(\text{theta2}\left(t\right)\right)\right)L1\,L2-$$
$$2\,m2\left(\frac{d}{d\,t}\,\text{theta1}\left(t\right)\right)^2\left(\cos\left(\text{theta1}\left(t\right)\right)\sin\left(\text{theta2}\left(t\right)\right)-\sin\left(\text{theta1}\left(t\right)\right)\cos\left(\text{theta2}\left(t\right)\right)\right)L1\,L2-$$
$$\left.2\,g\,m2\sin\left(\text{theta2}\left(t\right)\right)\,L2\right)/4$$

Combine trigonometric terms

```
(%i35) eq1:trigreduce(eq1);

       eq2:trigreduce(eq2);
```

(%o35)
$$\left(2\,m2\left(\frac{d}{d\,t}\,\text{theta2}\left(t\right)\right)^2\sin\left(\text{theta2}\left(t\right)-\text{theta1}\left(t\right)\right)\,L1\,L2-2\,m2\left(\frac{d^2}{d\,t^2}\,\text{theta2}\left(t\right)\right)\cos\left(\text{theta2}\left(t\right)-\text{thet}\right.\right.$$

$$4\,m2\left(\frac{d^2}{d\,t^2}\,\text{theta1}\,(t)\right)\,L1^2-m1\left(\frac{d^2}{d\,t^2}\,\text{theta1}\,(t)\right)\,L1^2-4\,g\,m2\sin\left(\text{theta1}\,(t)\right)\,L1-$$
$$2\,g\,m1\sin\left(\text{theta1}\,(t)\right)\,L1-4\,Ig1\left(\frac{d^2}{d\,t^2}\,\text{theta1}\,(t)\right))/\,4$$

(%o36) $$\left(-m2\left(\frac{d^2}{d\,t^2}\,\text{theta2}\,(t)\right)\,L2^2-2\,m2\left(\frac{d}{d\,t}\,\text{theta1}\,(t)\right)^2\sin\left(\text{theta2}\,(t)-\text{theta1}\,(t)\right)\,L1\,L2-\right.$$
$$2\,m2\left(\frac{d^2}{d\,t^2}\,\text{theta1}\,(t)\right)\cos\left(\text{theta2}\,(t)-\text{theta1}\,(t)\right)\,L1\,L2-2\,g\,m2\sin\left(\text{theta2}\,(t)\right)\,L2-$$
$$\left.4\,Ig2\left(\frac{d^2}{d\,t^2}\,\text{theta2}\,(t)\right)\right)/\,4$$

First differential equation - collect terms again

```
(%i37) eq1:facsum(eq1,diff(theta2(t),t,2),diff(theta1(t),t,2),diff(theta1(t),t),diff(theta2(
```

(%o37) $$\left(2\,m2\left(\frac{d}{d\,t}\,\text{theta2}\,(t)\right)^2\sin\left(\text{theta2}\,(t)-\text{theta1}\,(t)\right)\,L1\,L2-2\,m2\left(\frac{d^2}{d\,t^2}\,\text{theta2}\,(t)\right)\cos\left(\text{theta2}\,(t)-\text{thet}\right.\right.$$
$$\left.\left(\frac{d^2}{d\,t^2}\,\text{theta1}\,(t)\right)\left(4\,m2\,L1^2+m1\,L1^2+4\,Ig1\right)-2\,g\,\left(2\,m2+m1\right)\sin\left(\text{theta1}\,(t)\right)\,L1\right)/\,4$$

Second differential equation - collect terms again

```
(%i38) eq2:facsum(eq2,diff(theta2(t),t,2),diff(theta1(t),t,2),diff(theta1(t),t),diff(theta2(
```

(%o38) $$\left(-\left(\frac{d^2}{d\,t^2}\,\text{theta2}\,(t)\right)\,\left(m2\,L2^2+4\,Ig2\right)-2\,m2\left(\frac{d}{d\,t}\,\text{theta1}\,(t)\right)^2\sin\left(\text{theta2}\,(t)-\text{theta1}\,(t)\right)\,L1\,L2-\right.$$
$$\left.2\,m2\left(\frac{d^2}{d\,t^2}\,\text{theta1}\,(t)\right)\cos\left(\text{theta2}\,(t)-\text{theta1}\,(t)\right)\,L1\,L2-2\,g\,m2\sin\left(\text{theta2}\,(t)\right)\,L2\right)/\,4$$